Journal of Statistical Physics, Vol. 124, No. 1, July 2006 (© 2006)
DOI: 10.1007/s10955-006-9157-0

A Stationary Phase Approach to the Weak Coupling Schrödinger Equation

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Received March 16, 2006; accepted June 8, 2006
Published Online: July 12, 2006

We study the dynamics of a quantum particle governed by a linear Schrödinger equation with a scaled Gaussian potential. In the weak coupling limit the average dynamics of such a particle can be described by a linear Boltzmann equation. In this work we prove a bound for the rate at which the average dynamics of the quantum particle approach linear Boltzmann equation dynamics. For the so called simple diagrams, we use a stationary phase approach to establish an asymptotic expansion that provides the bound. Our stationary phase approach also provides a simple, formal method for computing the Boltzmann limit. Our work uses and extends results developed by L. Erdös and H.T. Yau.

KEY WORDS: weak coupling, Schrödinger, stationary phase

1. INTRODUCTION

If a quantum particle moves through a potential that varies on a much longer scale than the particle's wavelength then the dynamics of the particle are approximated by the Hamilton-Jacobi equations of classical particle motion.[(5)] This well known result allows one to recover Newtonian mechanics from quantum mechanics. A more difficult question is to describe the dynamics of a quantum particle traveling through a potential that varies on the same scale as the particle itself.

When the potential varies on the same scale as a quantum particle we often have a stochastic rather than a deterministic description of the potential. For instance, the potential within a semiconductor can be modeled by a random potential. We are led then to ask the following question: what are the dynamics of

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a quantum particle traveling through a random potential that varies on the same scale as the particle itself?

The weak coupling Schrödinger equation models a quantum particle moving through a weak random potential that varies on the same scale as the particle. In Refs. 3 and 8 it was shown that the average dynamics of such a particle can be characterized by a linear Boltzmann equation. In Refs. 1 a similar result was shown for the case of a time varying potential. The linear Boltzmann equation does not describe the motion of a classical particle moving through a potential. In this sense, the results of Refs. 3 and 8 give us a situation in which the dynamics of a quantum particle cannot be reduced to Newtonian dynamics.

In this work we derive a bound for the difference between the average quantum particle dynamics of the weak coupling Schrödinger equation and the dynamics of the linear Boltzmann equation. We prove our bound through an asymptotic expansion based on a stationary phase analysis. We apply our stationary phase analysis only to the simple diagrams discussed in Refs. 3. In principle one should be able to establish an expansion for the non-simple diagrams of Refs. 3, but our result does not require this since the simple diagrams carry the Boltzmann limit. Beyond the technical nature of our result, we feel that our analysis provides a valuable formalism in understanding the Boltzmann limit. Our stationary phase analysis provides an intuitive, formal method for understanding the weak coupling limit.

1.1. The Weak Coupling Problem

We study the weak coupling Schrödinger equation in the following scaling:

$$i\epsilon\psi_t(t,x) + \frac{1}{2}\epsilon^2\Delta\psi(t,x) - \sqrt{\epsilon}V\left(\frac{x}{\epsilon}\right)\psi(t,x) = 0; \tag{1.1}$$

where we assume that V is a stationary Gaussian field. We let $R(x)$ be the covariance function of V.

$$R(x) = E[V(y)V(x+y)]. \tag{1.2}$$

We assume that R satisfies the following conditions:

R is spherically symmetric;
For $j = 0, \ldots, 8, \quad i = 1, 2, 3, \ \partial_i^j \hat{R}(p_1, p_2, p_3)$ exists;
For $j = 0, \ldots, 8, \quad i = 1, 2, 3, \ \int_{\Re^d} dp(1 + |p|)|\partial_i^j \hat{R}(p_1, p_2, p_3)| \leq \infty$.

We let d be the spatial dimension of the weak coupling equation. We consider only the case $d = 3$. We take our initial data to be of WKB form. More specifically, let $h, S \in \mathcal{S}(\Re^d)$, where $\mathcal{S}(\Re^d)$ is Schwartz space. Then the initial data is of the following form:

$$\psi(0, x) = h(x) \exp\left[\frac{iS(x)}{\epsilon}\right] \tag{1.3}$$

We want to understand the average dynamics of the wave function as $\epsilon \to 0$. We will refer to the limit $\epsilon \to 0$ as the weak coupling limit.

It is well known that the dynamics of the linear Schrödinger equation are best understood in position-momentum phase space. A tool used in describing the phase space density of a wave function is the Wigner Transform.[4,6] The Wigner Transform transforms the wave function $\psi(t, x)$ into a phase space function $W_\psi(t, x, p)$.

$$W_\psi(t, x, p) = \frac{1}{(2\pi)^d} \int_{\Re^d} dy e^{ip\cdot y} \psi\left(t, x - \frac{y}{2}\right) \bar{\psi}\left(t, x + \frac{y}{2}\right) \tag{1.4}$$

or equivalently

$$W_\psi(t, x, p) = \frac{1}{(2\pi)^d} \int_{\Re^d} dq e^{ix\cdot q} \hat{\psi}\left(t, p + \frac{q}{2}\right) \bar{\hat{\psi}}\left(t, p - \frac{q}{2}\right). \tag{1.5}$$

The Wigner transform preserves the position and momentum densities of the wave function. That is, we have the following identities for any wave function ψ:

$$|\psi(t, x)|^2 = \int dp W(t, x, p).$$

$$|\hat{\psi}(t, p)|^2 = \int dx W(t, x, p). \tag{1.6}$$

But one must be careful when using the Wigner transform because $W(t, x, p)$ is not positive pointwise and therefore may not be thought of as a density. The Wigner transform is best thought of as an element of $\mathcal{S}'(\Re^{2d})$. In fact, for $J \in \mathcal{S}(\Re^{2d})$ we have

$$|(W_\psi(t), J)| \leq \frac{1}{(2\pi)^{d/2}} \int dy \left| \sup_x \int dp \exp[ip\cdot y] J(x, p) \right| \|\psi(t)\|_2^2. \tag{1.7}$$

If we consider the Wigner transform under the scaling $\frac{1}{\epsilon^d} W(t, x, \frac{p}{\epsilon})$ and take the weak coupling limit, assuming nice initial data (which our WKB initial data satisfies), the limit exists in $\mathcal{S}'(\Re^{2d})$ and it is a positive measure with total probability $\|\psi\|_2^2$.[4,6] This scaling is the appropriate scaling from which to examine the dynamics of the Schrödinger equation since the momentum probability density of a quantum particle is $\frac{1}{\epsilon^d} |\hat{\psi}(\frac{p}{\epsilon})|^2$.[5] We set then

$$W_\epsilon(t, x, p) = \frac{1}{\epsilon^d} W_\psi\left(t, x, \frac{p}{\epsilon}\right). \tag{1.8}$$

and we have that $\lim_{\epsilon\to 0} W_\epsilon(t, x, p)$ is a probability measure.

Using the Wigner transform, L. Erdös and H.T. Yau proved the following result:[(3)]

Theorem 1. *Let $B(t, x, p)$ be the solution of following linear Boltzmann equation in $C([0, \infty), \mathcal{S}'(\Re^d \times \Re^d))$:*

$$B_t(t, x, p) + p \cdot \nabla_x B(t, x, p) = \int dp' \sigma(p', p) B(t, x, p') - \Sigma(p) B(t, x, p)$$

$$B(0, x, p) = h^2(x)\delta(p - \nabla S(x)) \tag{1.9}$$

where

$$\sigma(p, q) = \hat{R}(p - q)\delta(p^2 - q^2).$$

$$\Sigma(p) = \int dq \sigma(p, q). \tag{1.10}$$

Then in $\mathcal{S}'(\Re^d \times \Re^d)$ with $d = 2$ or $d = 3$:

.

$$E[W_\epsilon(t, z, p)] \rightharpoonup B(t, z, p) \tag{1.11}$$

Our main result is the following error bound:

Theorem 2. *In the case of $d = 3$:*

$$|(E[W_\epsilon(t)], J) - (B(t), J)| \leq C(\delta, t, J)\epsilon^{\frac{1}{18} - \delta} \tag{1.12}$$

for any $\delta > 0$ and $J(x, p) \in \mathcal{S}(\Re^d \times \Re^d)$. The constant $C(\delta, t, J)$ is finite but depends on the values of t, δ, and J. As $\delta \to 0$, $C(\delta, t, J) \to \infty$.

We have two goals in this paper. First, we will use stationary phase expansions to formally analyze the so called simple diagrams of the weak coupling problem. Second, we will prove Theorem 2 by showing that our formal stationary phase expansions of the simple diagrams actually hold within the context of the weak coupling problem.

1.2. Stationary Phase Expansions

As will be explained in Sec. 2 the analysis of the weak coupling problem can be reduced to the computation of stationary phase integrals. A typical stationary phase integral of dimension n has the following form:

$$\int_\Omega dx \exp[i\phi(x)/\epsilon] f(x); \tag{1.13}$$

where $\Omega \subseteq \Re^n$, f and ϕ are real valued functions on $\Re^n$, and ϕ has critical points inside Ω. If f, ϕ, and Ω are sufficiently 'nice' then the stationary phase integral will have the following asymptotic expansion.[2] In order to make the formula simple we assume that x_0 is the only critical point of ϕ inside Ω.

$$\int_\Omega dx \exp[i\phi(x)/\epsilon] f(x)$$
$$= \frac{(2\pi\epsilon)^{n/2}}{|\nabla^2\phi(x_0)|^{1/2}} \exp\left[i\frac{\pi}{4} sig(\nabla^2\phi(x_0))\right] \exp[i\phi(x_0)/\epsilon] f(x_0) + O\left(\epsilon^{\frac{n+1}{2}}\right); \quad (1.14)$$

where for a matrix M, $sig(M)$ is the signature of M.

The basic technical goal of this paper is to justify and apply stationary phase asymptotic expansions to the integrals that arise in the weak coupling problem. The difficulty that we encounter is that the stationary phase integrals we consider are not 'nice' for three reasons.

First, if we use standard stationary phase arguments we require bounds on the nth derivative of f. This is fine for fixed n, but we require bounds for arbitrarily large n. We would require then some type of uniform bound on the derivatives of f. We cannot assume this for the form of f that we will need to analyze.

Second, our stationary phase integrals involve integration over complicated regions. Ω for us will be a complicated, non-smooth region. Although theoretically this is not an obstacle to applying standard stationary phase arguments, in practice we have found the computations to be very cumbersome.

Third, our stationary phase points are arbitrarily close to the boundary of Ω. In a standard stationary phase problem one works with a fixed set of critical points for any ϵ. This means that the critical points are either on the boundary of Ω or infinitely far from the boundary in terms of order ϵ. However, in our case for any given ϵ we will have to consider situations where the critical point is arbitrarily close or actually on the boundary. This is difficult to deal with especially within the context of a complicated, non-smooth boundary.

Our analysis of the stationary phase integrals encountered in the weak coupling problem will proceed along two lines. First, we want to develop a formal method for computing the stationary phase integrals. Second, we want to compute the stationary phase integrals rigorously and show that our formal assumptions lead to the correct results.

We make the following formal assumption:

FORMAL STATIONARY PHASE EXPANSION

Let x_0 be the only critical point of ϕ. If x_0 is in the interior of Ω then we assume that the following expansion holds:

$$\left| \int_\Omega dx \exp[i\phi(x)/\epsilon] f(x) - \frac{(2\pi\epsilon)^{n/2}}{|\nabla^2\phi(x_0)|^{1/2}} \exp\left[i\frac{\pi}{4} sig(\nabla^2\phi(x_0))\right] \exp[i\phi(x_0)/\epsilon] f(x_0) \right| \leq C^n \epsilon^{\frac{n+1}{2}} \int_\Omega dx |f(x)|. \tag{1.15}$$

where C is a constant independent of n, Ω, f, ϵ, *and* ϕ.

This assumption is not true for general Ω, f, and ϕ. However, the assumption will be true in the case of 'nice' Ω, f, and ϕ. In this sense our formal assumption reflects a belief that although our stationary phase integrals are non-standard, their asymptotic formulas are standard.

1.3. Notation

Before proceeding we set some notation and conventions that we use throughout the paper. We will use the constant C to describe any real number that does not change in the various limits that we will take and that does not depend on the time variable t. If $x \in \mathbb{C}$ then we let $\bar{x}$ represent the complex conjugate of x. $\mathcal{S}(\Re^m)$ represents the space of Schwartz functions on $\Re^m$.

The variables s_j, $s_{j,k}$, s'_j, and $s'_{j,k}$ will be used to represent time coordinates while p_j, $p_{j,k}$, p'_j, and $p'_{j,k}$ will be used as momentum variables. The variable x will be used to represent spacial coordinates. Unless otherwise noted, integrals over momentum and space variables are taken over all of $\Re^d$. For $f : \Re^m \to \mathbb{C}$ and $g : \Re^m \to \mathbb{C}$ we define:

$$\hat{f}(p) = \frac{1}{(2\pi)^{m/2}} \int_{\Re^m} dx \exp[-ix \cdot p] f(x),$$
$$(f, g) = \int_{\Re^m} dx \bar{f}(x) g(x). \tag{1.16}$$

We will encounter various nested time integrations. In order to compactify our formulas we introduce the following notation.

$$\int^{t,n} ds = \int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \dots \int_0^{s_{n-1}} ds_n.$$
$$\int^{t,n} ds' = \int_0^t ds'_1 \int_0^{s'_1} ds'_2 \int_0^{s'_2} ds'_3 \dots \int_0^{s_{n-1'}} ds'_n.$$
$$\Delta s_j = s_j - s'_j.$$
$$T_j = s_j - s_{j-1}, T'_j = s'_j - s'_{j-1}. \tag{1.17}$$

We also introduce the following notation for integrating the Δs_j variables over an arbitrary region.

$$\int_\Omega d\Delta s = \int_\Omega d\Delta s_1 d\Delta s_2 \dots d\Delta s_n \text{ where } \Omega \text{ is a region in } \Re^n. \tag{1.18}$$

When we deal with momentum integrations it will often be convenient to switch to spherical coordinates. For a momentum variable p_j we associate the variables r_j and $\hat{\mu}_j$ with the relationship

$$\hat{\mu}_j = \frac{p_j}{|p_j|}, r_j = |p_j|. \tag{1.19}$$

We let $S(1)$ be the unit sphere in $\Re^d$. Then we set the following notation for an iterated collection of surface integrals.

$$\int^n d\hat{\mu} = \int_{S(1)} d\hat{\mu}_0 \int_{S(1)} d\hat{\mu}_1 \dots \int_{S(1)} d\hat{\mu}_{n-1}. \tag{1.20}$$

2. THE FEYNMAN DIAGRAM EXPANSION

In this section we explain the basic method that was used in Refs. 3 to analyze the weak coupling Schrödinger problem. We include this discussion for the sake of clarity and completeness. Recall that our main goal is to compute an asymptotic expansion in ϵ of the following expression:

$$\int dpdx J(x,p) E[W_\epsilon(t,x,p)] \tag{2.1}$$

We will compute this quantity by expressing it as a sum of Feynman diagrams. We will then compare this sum of Feynman diagrams to the Born expansion of the Boltzmann equation and show that the two are the same up to a small error that we can control.

In this section we first recall the Born expansion solution of the Boltzmann equation. We then show how to expand the wave function $\psi(t,x)$ in a Born series. Finally we use this Born expansion to develop an expansion for the Wigner transform of $\psi(t,x)$.

2.1. Born Expansion of the Boltzmann Equation

We solve the Boltzmann equation given by (1.9) through a Born expansion. Set $s_0 = t$ and $s_{n+1} = 0$. Define $B_n(t,x,p)$ as follows:

$$B_n(t,x,p_0) = \int^{t,n} ds \int dp_1 \dots dp_n U_n h^2(x_n) \delta(p_n - \nabla S(x_n)); \tag{2.2}$$

where

$$U_n = \left(\prod_{k=0}^{n-1} \exp[-\Sigma(p_k)(s_k - s_{k+1})]\sigma(p_k, p_{k+1})\right) \exp[-\Sigma(p_n)s_n],$$

$$x_n = x - \sum_{j=0}^{n} p_j(s_j - s_{j+1}). \tag{2.3}$$

B_n is the nth term in the Born expansion of the Boltzmann equation. The Born series converges. In fact we have the following lemmas.

Lemma 2.1. *Let* $J(x, p) \in \mathcal{S}(\Re^d \times \Re^d)$. *Then*

$$|(B_n(t), J)| \leq \frac{(Ct)^n}{n!}\|J\|_\infty; \tag{2.4}$$

where the constant C is independent of n.

Lemma 2.2. *Let*

$$B(t, x, p) = \sum_{n=0}^{\infty} B_n(t, x, p). \tag{2.5}$$

Then $B(t, x, p)$ *solves the Boltzmann equation given by (1.9) in* $C([0, \infty), \mathcal{S}'(\Re^d \times \Re^d))$.

2.2. The Born Expansion of ψ

Let $G_0(t, x)$ be the free evolution Green's function for the Schrödinger equation in the same scaling as (1.1), the weak coupling Schrödinger equation. That is,

$$i\epsilon\partial_t G_0(t, x) + \frac{1}{2}\epsilon^2 \Delta G_0(t, x) = 0.$$

$$G_0(0, x) = \delta(x). \tag{2.6}$$

We can write out the Born expansion of $\hat{\psi}(t, p_0)$ up to order N. Let $\hat{\psi}(p) = \hat{\psi}(0, p)$. Set for $n > 0$,

$$\hat{\psi}_n\left(t, \frac{p_0}{\epsilon}\right) = \left(\frac{-i}{\sqrt{\epsilon}}\right)^n \int^{t,n} ds \int dp_1 \ldots dp_n G_0\left(t - s_1, \frac{p_0}{\epsilon}\right) \hat{V}(p_0 - p_1)$$
$$\times G_0\left(s_1 - s_2, \frac{p_1}{\epsilon}\right) \hat{V}(p_1 - p_2) \ldots G_0\left(s_{n-1} - s_n, \frac{p_{n-1}}{\epsilon}\right)$$
$$\times \hat{V}(p_{n-1} - p_n) G_0\left(s_n, \frac{p_n}{\epsilon}\right) \hat{\psi}\left(\frac{p_n}{\epsilon}\right). \tag{2.7}$$

For $n = 0$ set

$$\hat{\psi}_0\left(t, \frac{p_0}{\epsilon}\right) = G_0\left(t, \frac{p_0}{\epsilon}\right)\hat{\psi}\left(\frac{p_0}{\epsilon}\right) \tag{2.8}$$

and for the remainder term in the Born expansion define Ψ_n as follows:

$$\Psi_n\left(t, \frac{p_0}{\epsilon}\right) = \frac{-i}{\sqrt{\epsilon}}\int_0^t ds \int dp_1 G\left(t-s, \frac{p_0}{\epsilon}\right)\hat{V}(p_0 - p_1)\hat{\psi}_n\left(s, \frac{p_1}{\epsilon}\right); \tag{2.9}$$

where $G(t, p)$ is the Green's function of the weak coupling equation. We then have the following expansion,

$$\hat{\psi}\left(t, \frac{p_0}{\epsilon}\right) = \sum_{j=0}^{N}\hat{\psi}_j\left(t, \frac{p_0}{\epsilon}\right) + \Psi_N\left(t, \frac{p_0}{\epsilon}\right) \tag{2.10}$$

Following,[3] we represent a typical term, $\hat{\psi}_j(t, \frac{p_0}{\epsilon})$, in the Born expansion by a diagram as shown in Fig. 1. Each edge in the diagram is associated with a momentum. Each vertex in the diagram is associated with a potential and a time found in the expansion given by (2.7). In order to make our formulas and diagrams more readable we introduce the following notation for the potentials.

$$\begin{aligned}\hat{V}_{(j,T)} &= \hat{V}(p_j - p_{j+1})\\ \hat{V}_{(j,B)} &= \bar{\hat{V}}(p'_j - p'_{j+1})\end{aligned} \tag{2.11}$$

The subscripts T and B stand for top and bottom respectively. This nomenclature will make sense when we form Feynman diagrams in the next subsection.

We can express $\bar{\hat{\psi}}_k(t, \frac{p'_0}{\epsilon})$ as a diagram just as we did for $\hat{\psi}_j(t, \frac{p_0}{\epsilon})$. We will use prime notation for the case of a complex conjugated wave function as shown in Fig. 2.

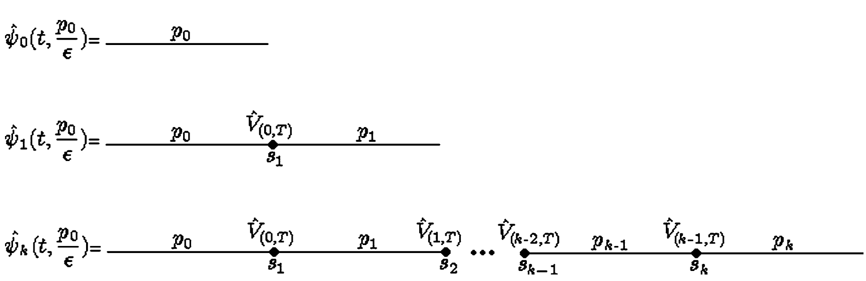

Fig. 1. Diagrams for Born expansion terms.

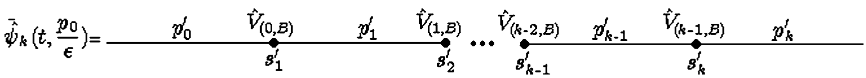

Fig. 2. Diagram for complex conjugate Born term.

2.3. The Wigner Transform Expansion

The evaluation of the Wigner transform against a test function, Eq. (2.1), can now be expanded through the Born series of ψ. First we notice the following equality which follows from a Plancherel argument in x on (2.1) and the use of (1.5).

$$\int dpdx J(x,p)E\left[W_\epsilon\left(t,x,\frac{p}{\epsilon}\right)\right] = \frac{1}{(2\pi)^{d/2}}\int dpd\zeta\,\bar{\hat{J}}(\zeta,p)\frac{1}{\epsilon^d}E\left[\hat{\psi}\left(t,\frac{p}{\epsilon}+\frac{\zeta}{2}\right)\bar{\hat{\psi}}\left(t,\frac{p}{\epsilon}-\frac{\zeta}{2}\right)\right] \quad (2.12)$$

This equality allows us to plug in our Born expansion for $\psi(t)$ and arrive at the following expansion for (2.1).

$$\int dpdx J(x,p)\frac{1}{\epsilon^d}E\left[W_{\psi(t)}\left(x,\frac{p}{\epsilon}\right)\right] = \mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3. \quad (2.13)$$

Where we set

$$\mathcal{L}_1 = \frac{1}{(2\pi)^{d/2}}\sum_{j=0}^{N}\sum_{k=0}^{N}\int dpd\zeta\,\bar{\hat{J}}(\zeta,p)\frac{1}{\epsilon^d}E\left[\hat{\psi}_j\left(t,\frac{p}{\epsilon}+\frac{\zeta}{2}\right)\bar{\hat{\psi}}_k\left(t,\frac{p}{\epsilon}-\frac{\zeta}{2}\right)\right],$$

$$\mathcal{L}_2 = \frac{1}{(2\pi)^{d/2}}\sum_{j'=0}^{N}\int dpd\zeta\,\bar{\hat{J}}(\zeta,p)\frac{1}{\epsilon^d}2Re\left(E\left[\hat{\psi}_{j'}\left(t,\frac{p}{\epsilon}+\frac{\zeta}{2}\right)\bar{\Psi}_N\left(t,\frac{p}{\epsilon}-\frac{\zeta}{2}\right)\right]\right),$$

$$\mathcal{L}_3 = \frac{1}{(2\pi)^{d/2}}\int dpd\zeta\,\bar{\hat{J}}(\zeta,p)\frac{1}{\epsilon^d}E\left[\Psi_N\left(t,\frac{p}{\epsilon}+\frac{\zeta}{2}\right)\bar{\Psi}_N\left(t,\frac{p}{\epsilon}-\frac{\zeta}{2}\right)\right]. \quad (2.14)$$

We will eventually show that $\mathcal{L}_1$ gives the Boltzmann limit while $\mathcal{L}_2$ and $\mathcal{L}_3$ can be treated as error terms.

2.3.1. Feynman Diagrams

In order to evaluate the Wigner transform against a test function we see from (2.13) that we must be able to compute expressions of the following form:

$$E\left[\hat{\psi}_j\left(t,\frac{p_0}{\epsilon}\right)\bar{\hat{\psi}}_k\left(t,\frac{p'_0}{\epsilon}\right)\right]. \quad (2.15)$$

In computing $E[\hat{\psi}_j(t)\bar{\hat{\psi}}_k(t)]$ the expectation is applied to the potential terms $\hat{V}$ in (2.7). Associated with the term $\hat{\psi}_j(t)$ we have the potential factors: $\prod_{h=0}^{j-1}\hat{V}_{(h,T)}$. Associated with the term $\bar{\hat{\psi}}_k(t)$ we have the potential factors: $\prod_{h'=0}^{k-1}\hat{V}_{(h',B)}$. We then need to consider the expression:

$$E\left[\prod_{h=0}^{j-1}\hat{V}_{(h,T)}\prod_{h'=0}^{k-1}\hat{V}_{(h',B)}\right]. \tag{2.16}$$

We use the Gaussian structure of V to compute this expression explicitly. We have $j+k$ potentials. Since V is Gaussian if $j+k$ is odd then the above expectation evaluates to zero. So we assume that $j+k$ is even. Let π represent a pairing of the potentials. More precisely π satisfies the following criteria.

$$\begin{aligned}
\text{Let } E_{j,k} &= \{(0,T),(1,T),\ldots,(j-1,T),(0,B),(1,B),\ldots,(k-1,B)\}.\\
\pi : E_{j,k} &\to E_{j,k}.\\
\pi(\pi) &= i.\\
\text{For } e &\in E_{j,k}, \pi(e) \neq e.
\end{aligned} \tag{2.17}$$

Our intuition here is that π pairs the potential $\hat{V}_e$ with $\hat{V}_{\pi(e)}$. Wick's Theorem gives us the following:

$$E\left[\prod_{h=0}^{j-1}\hat{V}_{h,T}\prod_{h'=0}^{k-1}\bar{\hat{V}}_{h',B}\right] = \sum_{\pi}\left(\prod_{e\in E_{j,k}} E[\hat{V}_e\hat{V}_{\pi(e)}]\right)^{1/2}. \tag{2.18}$$

We then define an expectation associated with a given pairing.

$$E^{\pi}\left[\prod_{h=0}^{j-1}\hat{V}_{h,T}\prod_{h'=0}^{k-1}\bar{\hat{V}}_{h',B}\right] = \left(\prod_{e\in E_{j,k}} E[\hat{V}_e\hat{V}_{\pi(e)}]\right)^{1/2} \tag{2.19}$$

and we may write

$$E\left[\hat{\psi}_j\left(t,\frac{p_0}{\epsilon}\right)\bar{\hat{\psi}}_k\left(t,\frac{p_0'}{\epsilon}\right)\right] = \sum_{\pi} E^{\pi}\left[\hat{\psi}_j\left(t,\frac{p_0}{\epsilon}\right)\bar{\hat{\psi}}_k\left(t,\frac{p_0'}{\epsilon}\right)\right]. \tag{2.20}$$

Following,[3] we associate with each expectation $E^{\pi}[\hat{\psi}_j(t,\frac{p_0}{\epsilon})\bar{\hat{\psi}}_k(t,\frac{p_0'}{\epsilon})]$ a Feynman diagram. The Feynman diagram consists of two diagrams representing the Born expansions of $\hat{\psi}_j$ and $\bar{\hat{\psi}}_k$ connected by dashed lines. The dashed lines are referred to as pairing lines and connect the vertices associated with the potentials that are paired. Figure 3 represents the Feynman diagram corresponding to $E^{\pi}[\hat{\psi}_2(t,\frac{p_0}{\epsilon})\bar{\hat{\psi}}_2(t,\frac{p_0'}{\epsilon})]$ with $\pi((0,T)) = \pi((1,B))$ and $\pi((1,T)) = \pi((0,B))$.

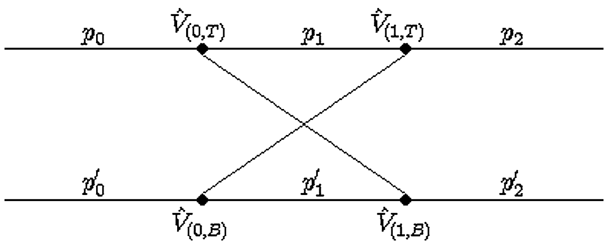

Fig. 3. Feynman Diagram.

We classify our Feynman diagrams into three groups. When $j = k$ and $\pi((m, T)) = (m, B)$ then we refer to the diagram as a **ladder diagram**. For $j = k = 2$, Figure 4 shows the Feynman diagram that is a ladder diagram.

For general j, k a Feynman diagram is said to be a **simple diagram** if $\exists L$ such that $\exists (h_1 < h_2 < \ldots < h_L), \exists (h'_1 < h'_2 < \ldots < h'_L)$ for which the following three conditions hold.

1. $\pi((h_m, T)) = (h'_m, B)$
2. $m \notin (h_1, \ldots, h_L) \Longrightarrow \pi((m, T)) = (m + 1, T)$ or $\pi((m, T)) = (m - 1, T)$
3. $m' \notin (h'_1, \ldots, h'_L) \Longrightarrow \pi((m', B)) = (m' + 1, B)$ or $\pi((m', B)) = (m' - 1, B)$

This involved definition is graphically very simple. A simple diagram consists of a collection of non-crossing pairing lines, which we refer to as **ladder rungs**, that connect the two wave function diagrams; between these ladder rungs each vertex is connected to one of its neighbors. Figure 5 shows a simple diagram for the case $j = 6, k = 4$.

Ladder diagrams are a type of simple diagram. Any diagram that is not simple we refer to as a **non-simple diagram**. The important characteristic of non-simple diagrams is that they contain pairing lines that cross or pairing lines nested within each other. This characteristic was exploited by Erdös and Yau to show that non-simple diagrams have small contribution to (2.1).[3]

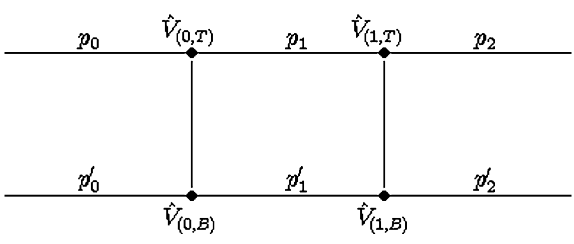

Fig. 4. Ladder Diagram for $j = k = 2$.

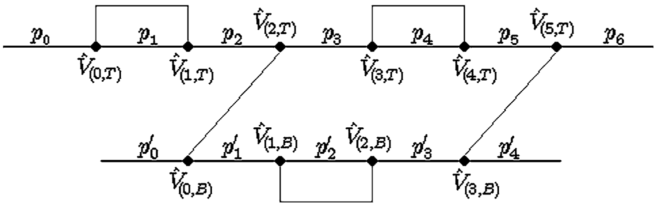

Fig. 5. Simple Diagram for $j = 6$, $k = 4$.

3. LADDER DIAGRAMS

In this section we analyze ladder diagrams. We first write down an explicit formula for ladder diagrams. We then use our formal stationary phase assumption to write down an asymptotic expansion for the ladder diagrams. Finally, we show that our formal asymptotic expansion holds rigorously.

3.1. Ladder Diagram Formula

We want to write down an explicit formula for ladder diagrams. That is, we want formulas for the following expression:

$$E^{\pi}\left[\hat{\psi}_n\left(t, \frac{p_0}{\epsilon}\right)\bar{\hat{\psi}}_n\left(t, \frac{p'_0}{\epsilon}\right)\right]; \tag{3.1}$$

where π is a ladder diagram.

We can easily evaluate the expectation of the potentials, given by (2.19), that is associated with the ladder diagram pairing. Set $p_0 - p'_0 = \zeta$. Then

$$\begin{aligned} E^{\pi}\left[\prod_{h=0}^{n-1}\hat{V}_{h,T}\prod_{h'=0}^{n-1}\bar{\hat{V}}_{h',B}\right] &= \prod_{h=0}^{n-1} E\big[\hat{V}(p_h - p_{h+1})\bar{\hat{V}}(p'_h - p'_{h+1})\big] \\ &= \prod_{h=0}^{n-1}\hat{R}(p_h - p_{h+1})\delta((p_h - p_{h+1}) - (p'_h - p'_{h+1})) \\ &= \prod_{h=1}^{n}\hat{R}(p_{h-1} - p_h)\delta((p_h - p'_h) - \zeta). \end{aligned} \tag{3.2}$$

Using this formula we can easily find an expression for the n rung ladder diagram.

$$\begin{aligned} &E\left[\hat{\psi}_n\left(t, \frac{p_0}{\epsilon}\right)\bar{\hat{\psi}}_n\left(t, \frac{p'_0}{\epsilon}\right)\right] \\ &\quad = \frac{1}{\epsilon^n}\int^{t,n} ds \int^{t,n} ds' \int dp_1 \ldots dp_n \exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right] f_{LD}; \end{aligned} \tag{3.3}$$

where

$$f_{LD} = \exp\left[\frac{-i\zeta}{2}\sum_{j=1}^{n} p_j(T_j + T_j')\right]\prod_{h=0}^{n-1}\hat{R}(p_h - p_{h+1})\hat{\psi}\left(\frac{p_n}{\epsilon}+\frac{\zeta}{2}\right)\bar{\hat{\psi}}\left(\frac{p_n}{\epsilon}-\frac{\zeta}{2}\right) \tag{3.4}$$

and

$$\begin{aligned}&\phi_{LD}(p_0, p_1, \ldots, p_n, \Delta s_1, \ldots, \Delta s_n)\\ &\quad = -p_0^2\Delta s_1 + p_1^2(\Delta s_1 - \Delta s_2) + \ldots + p_{n-1}^2(\Delta s_{n-1} - \Delta s_n) + p_n^2\Delta s_n.\end{aligned} \tag{3.5}$$

Recall that we are interested in evaluating the Wigner transform against a test function. In order to do this using the Wigner transform expansion we must evaluate the different diagrams against a test function. In this section we are interested in evaluating ladder diagrams against a test function. Specifically we want to compute an expansion for the following quantity which we label $L_n(t)$:

$$L_n(t) = \int dp_0 d\zeta\, \bar{\hat{J}}(\zeta, p_0)\frac{1}{\epsilon^d}E^{\pi}\left[\hat{\psi}_n\left(t, \frac{p_0}{\epsilon}+\frac{\zeta}{2}\right)\bar{\hat{\psi}}_n\left(t, \frac{p_0}{\epsilon}-\frac{\zeta}{2}\right)\right]; \tag{3.6}$$

where π is the ladder diagram pairing with n rungs. Plugging in our formula for a ladder diagram of length n, Eq. (3.3), we arrive at the following:

$$L_n(t) = \frac{1}{\epsilon^d}\frac{1}{\epsilon^n}\int^{t,n} ds \int^{t,n} ds' \int dp_0 \ldots dp_n \int d\zeta\, \bar{\hat{J}}(\zeta, p_0)\exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right]f_{LD} \tag{3.7}$$

The case $n = 0$ is a special case. Through explicit computations we have the following identity.

$$\begin{aligned}&\int dp_0 d\zeta\, \bar{\hat{J}}(\zeta, p_0)\frac{1}{\epsilon^d}\hat{\psi}_0\left(t, \frac{p_0}{\epsilon}+\frac{\zeta}{2}\right)\bar{\hat{\psi}}_0\left(t, \frac{p_0}{\epsilon}-\frac{\zeta}{2}\right)\\ &\quad = \frac{1}{\epsilon^d}\int dp dx\, J(x, p) W\left(0, x - pt, \frac{p}{\epsilon}\right).\end{aligned} \tag{3.8}$$

Keeping this identity in mind, in our analysis of $L_n(t)$ we can assume that $n > 0$.

3.2. Formal Stationary Phase Expansion for Ladder Diagrams

We start our analysis of $L_n(t)$ by using formal stationary phase arguments. We reexpress (3.7) as follows:

$$\begin{aligned}L_n(t) = \frac{1}{\epsilon^d}\frac{1}{\epsilon^n}\int^{t,n} ds \int dp_n \int d\zeta\bigg(\int_{\Omega_n} d\Delta s \int dp_0 \ldots dp_{n-1}\hat{J}(\zeta, p_0)\\ \times \exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right]f_{LD}\bigg);\end{aligned} \tag{3.9}$$

where Ω_n is a region in $\Re^n$ defined, through $\int_{\Omega_n} d\Delta s$, as follows:

$$\int_{\Omega_n} d\Delta s = \int_{s_1-t}^{s_1} d\Delta s_1 \int_{\Delta s_1+(s_2-s_1)}^{s_2} d\Delta s_2 \ldots \int_{\Delta s_{n-1}+(s_n-s_{n-1})}^{s_n} d\Delta s_n. \tag{3.10}$$

We need to apply our formal stationary phase argument to the expression in parenthesis in (3.9). Our first step will be to compute the stationary phase points of ϕ_{LD}. This gives us the following stationary phase surface:

$$\begin{aligned} \Delta s_1 = \Delta s_2 = \cdots = \Delta s_n = 0. \\ p_0^2 = p_1^2 = \cdots = p_n^2. \end{aligned} \tag{3.11}$$

We recall our notation $p_j = r_j \hat{\mu}_j$. In these coordinates we can integrate through the stationary phase surface and be left with a stationary phase point.

$$L_n(t) = \frac{1}{\epsilon^d} \frac{1}{\epsilon^n} \int^{t,n} ds \int dp_n \int d\zeta \int^n d\hat{\mu} Z; \tag{3.12}$$

with Z given as follows:

$$\begin{aligned} &Z(s_1, \ldots, s_n, \hat{\mu}_0, \ldots, \hat{\mu}_{n-1}, p_n) \\ &= \int_{\Omega_n} d\Delta s \int dr_0 dr_1 \ldots dr_{n-1} r_0^{d-1} \ldots r_{n-1}^{d-1} \bar{\hat{J}}(\zeta, p_0) \exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right] f_{LD}. \end{aligned} \tag{3.13}$$

We can now apply our formal stationary phase expansions to compute Z. The integral is a $2n$ dimensional stationary phase integral. The stationary phase point of ϕ, which we label x_{cp}, in the variables $\Delta s_1, \ldots, \Delta s_n, r_0, \ldots, r_{n-1}$ is given by the following coordinates:

$$\begin{aligned} \Delta s_1 = \Delta s_2 = \cdots = \Delta s_n = 0. \\ r_0 = r_1 = \cdots = r_{n-1} = |p_n|. \end{aligned} \tag{3.14}$$

In (3.13) above, x_{cp} is a function of p_n. As long as $|p_n| \neq 0$ then x_{cp} is in the interior of the region of integration $\Omega_n \times (\Re^+)^n$. The case $|p_n| = 0$ has Lebesgue measure zero in (3.12), so it will have no effect.

Through explicit computation we have

$$\begin{aligned} \det(\nabla^2 \phi(x_{cp})) = |p_n|^{2n}; \\ \text{signature}(\nabla^2 \phi(x_{cp})) = 0. \end{aligned} \tag{3.15}$$

Finally, we have

$$\begin{aligned} &\int_{\Omega_n} d\Delta s \int dr_0 dr_1 \ldots dr_{n-1} r_0^{d-1} \ldots r_{n-1}^{d-1} \left|\bar{\hat{J}}(\zeta, p_0) f_{LD}\right| \\ &\le (Ct)^n \left|\psi\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right)\right| \sup_p \left|\hat{J}(\zeta, p)\right|. \end{aligned} \tag{3.16}$$

We can now apply our formal stationary phase assumptions to arrive at the bound:

$$|Z - Z_{cp}| \leq (Ct)^n \epsilon^{n+\frac{1}{2}} \left| \psi\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right) \right| \sup_p \left| \bar{\hat{J}}(\zeta, p) \right|; \tag{3.17}$$

where Z_{cp} is given by

$$\begin{aligned} Z_{cp} &= (2\pi |p_n|\epsilon)^n f_{LD}(x_{cp}) \\ &= \hat{J}(\zeta, |p_n|\hat{\mu}_0) \exp\left[-i|p_n|\zeta \cdot \sum_{j=1}^{n} \hat{\mu}_j (s_j - s_{j+1}) \right] (2\pi |p_n|)^n \\ &\quad \times \prod_{h=0}^{n-1} \hat{R}(|p_n|(\hat{\mu}_h - \hat{\mu}_{h+1})) \hat{\psi}\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right) \end{aligned} \tag{3.18}$$

Plugging this result into our expression for $L_n(t)$ given by (3.12) we have,

$$\left| L_n(t) - \frac{1}{\epsilon^d} \int^{t,n} ds \int dp_n \int d\zeta \, Z_{cp} \right| \leq (Ct^2)^n \frac{\sqrt{\epsilon}}{n!}. \tag{3.19}$$

Now we use the Wigner transform formula given by Eq. 1.5 and apply Plancherel in ζ to arrive at the following bound.

$$\begin{aligned} &\left| L_n(t) - (2\pi)^{d/2} \int^{t,n} ds \int dp_n \int dx \int d\hat{\mu} J(x_0, |p_n|\hat{\mu}_0) \right. \\ &\left. \times (2\pi |p_n|)^n \prod_{h=0}^{n-1} \hat{R}(|p_n|(\hat{\mu}_h - \hat{\mu}_{h+1})) W_\epsilon(0, x, p_n) \right| \leq (Ct^2)^n \frac{\sqrt{\epsilon}}{n!}; \end{aligned} \tag{3.20}$$

where

$$x_0 = x + |p_n| \sum_{j=1}^{n} \hat{\mu}_j (s_j - s_{j+1}) \tag{3.21}$$

3.3. Proof of Stationary Phase Expansion for Ladder Diagram

Our goal in this section is to prove our formal result for $L_n(t)$ given by (3.20). In order to prove this stationary phase result we introduce some new notation.

$$\begin{aligned} \tilde{f}_{LD} &= \bar{\hat{J}}(\zeta, p_0) \prod_{h=0}^{n-1} \hat{R}(p_h - p_{h+1}) \hat{\psi}\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right). \\ K_n &= \int d\zeta \int dp_0 \ldots dp_n \exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right] \exp\left[\frac{-i\zeta}{2} \sum_{j=1}^{n} p_j (T_j + T_j')\right] \tilde{f}_{LD} \end{aligned} \tag{3.22}$$

Using this notation we have the following compact form for $L_n(t)$:

$$L_n(t) = \frac{1}{\epsilon^d}\frac{1}{\epsilon^n}\int^{t,n} ds \int_{\Omega_n} d\Delta s K_n \tag{3.23}$$

We now proceed to prove our stationary phase result.

3.3.1. Bounding K_n

Our first goal will be to bound K_n as a function of the time variables. Specifically, we want to prove the following lemma.

Lemma 3.1

$$|K_n(\Delta s_1, \ldots, \Delta s_n)| \le C^n \epsilon^d \|\psi\|_2^2 S_n \tag{3.24}$$

where

$$S_j = \frac{1}{1 + |\frac{\Delta s_1}{\epsilon}|^{3/2}} \frac{1}{1 + |\frac{\Delta s_1 - \Delta s_2}{\epsilon}|^{3/2}} \cdots \frac{1}{1 + |\frac{\Delta s_{j-1} - \Delta s_j}{\epsilon}|^{3/2}} \tag{3.25}$$

In order to prove the above lemma we introduce the following Plancherel type lemma that is similar to a lemma of H. Spohn[8] but made specific to our case.

Lemma 3.2 *Let $\sigma^{(0,n-1)} : \{0, \ldots, n-1\} \to \{0, 1\}$. Let $H \in C^\infty(\Re^{nd})$ and let H and all its derivatives be in $L^1(\Re^{nd})$. Let $\zeta_j \in \Re^d$. Set $\zeta = (\zeta_0, \ldots, \zeta_{n-1})$ and $p = (p_0, \ldots, p_{n-1})$. Let $\sigma_j = \sigma(j)$. Then for $d = 3$ we have*

$$\left| \int dp_0 \ldots dp_{n-1} \exp(ip_0^2 v_0 + \ldots + ip_{n-1}^2 v_{n-1}) \exp(ip \cdot \zeta) H(p) \right|$$
$$\le C^n \frac{1}{1 + |v_0|^{3/2}} \cdots \frac{1}{1 + |v_{n-1}|^{3/2}} \int dp_0 \ldots dp_{n-1} \Upsilon_n(H)$$

where we define the operator Υ_n as follows:

$$\Upsilon_n(H) = \sum_{\sigma^{(0,n-1)}} \left| \left(\prod_{j=0}^{n-1} (\sigma_j + (1 - \sigma_j)) \Delta_{p_j}^2 \right) H \right|. \tag{3.26}$$

Notice that the linear exponential $\exp(ip \cdot \zeta)$ has no effect on the bound.

Proof of Lemma 3.2: We prove the lemma by induction. Set $n = 1$. If $v_0 < 1$ then we use the following trivial bound,

$$\left| \int dp_0 \exp[ip_0^2 v_0] \exp(ip_0 \cdot \zeta_0) H(p_0) \right| \le \int dp_0 |H(p_0)|. \tag{3.27}$$

Now take $v_0 > 1$. Then applying Plancherel we have

$$\left|\int dp_0 \exp[ip_0^2 v_0] \exp(ip_0 \cdot \zeta_0) H(p_0)\right| \leq \frac{(2\pi)^{3/2}}{v_0^{3/2}} \int dx_0 |\hat{H}(x_0 - \zeta_0)|$$

$$\leq \frac{(2\pi)^{3/2}}{v_0^{3/2}} \int dx_0 |\hat{H}(x_0)|$$

$$\leq \frac{(2\pi)^{3/2}}{v_0^{3/2}} \int_{|x_0|\leq 1} dx_0 |\hat{H}(x_0)| + \frac{(2\pi)^{3/2}}{v_0^{3/2}} \int_{|x_0|>1} dx_0 |\hat{H}(x_0)|$$

$$\leq \frac{(2\pi)^{3/2}}{v_0^{3/2}} \left[\frac{4\pi}{3} \int dp_0 |H(p_0)| + 4\pi \int dp_0 |\Delta_{p_0}^2 H(p_0)|\right]. \tag{3.28}$$

Taking these two cases together we have for any v_0:

$$\left|\int dp_0 \exp[ip_0^2 v_0] \exp(ip_0 \cdot \zeta_0) H(p_0)\right|$$

$$\leq 8\pi(2\pi)^{3/2} \frac{1}{1 + v_0^{3/2}} [\int dp_0 |H(p_0)| + \int dp_0 |\Delta_{p_0}^2 H(p_0)|]. \tag{3.29}$$

The proof now follows for arbitrary n by a simple inductive argument. □

We now make H specific to K_n by setting $H = \tilde{f}_{LD}$. We have the following lemma.

Lemma 3.3

$$\Upsilon_n(\tilde{f}_{LD}) \leq 2^n \sum_{j_0=0}^{4} |\nabla_{p_0}^{j_0} \bar{\tilde{J}}(\zeta, p_0)| \prod_{k=1}^{n-1} \sum_{j_k=0}^{8} |\nabla^{j_k} \hat{R}(p_{k-1} - p_k)|$$

$$\times |\hat{\psi}\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right)| \tag{3.30}$$

and

$$\int d\zeta \int dp_0 \dots dp_n \Upsilon_n(\tilde{f}_{LD}) \leq C^n \epsilon^d \|\psi\|_2^2. \tag{3.31}$$

Proof of Lemma 3.3: We prove the second inequality above. In the process we demonstrate the first inequality as well.

$$\sum_{\sigma^{(0,n-1)}} \int d\zeta \int dp_0 \dots dp_n | \left(\prod_{j=0}^{n-1} (\sigma_j + (1 - \sigma_j) \Delta_{p_j}^2)\right) \tilde{f}_{LD}|$$

$$\le 2^n \sup_{\sigma} \int d\zeta \int dp_0 \ldots dp_n | \left(\prod_{j=0}^{n-1} (\sigma_j + (1-\sigma_j)\Delta^2_{p_j}) \right) \tilde{f}_{LD} |$$

$$\le 2^n \int d\zeta \int dp_0 \ldots dp_n \sum_{j_0=0}^{4} |\nabla^{j_0}_{p_0} \bar{\hat{J}}(\zeta, p_0)| \prod_{k=1}^{n-1} \sum_{j_k=0}^{8} |\nabla^{j_k} \hat{R}(p_{k-1} - p_k)|$$

$$\times \left| \hat{\psi}\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right) \right|$$

$$\le 2^n C^{n-1} \int d\zeta \int dp_n \sup_{p_0} \sum_{j_0=0}^{4} |\nabla^{j_0}_{p_0} \bar{\hat{J}}(\zeta, p_0)| \left| \hat{\psi}\left(\frac{p_n}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_n}{\epsilon} - \frac{\zeta}{2}\right) \right|$$

$$\le 2^n C^n \epsilon^d \|\psi\|_2^2 \tag{3.32}$$

□

The proof of Lemma 3.1 now follows from an application of Lemmas 3.2 and 3.3.

3.3.2. Localizing Time Coordinates to Stationary Phase Point

We now show that the main contribution to our integral comes from the surface $\Delta s_1 = \Delta s_2 = \cdots = \Delta s_n = 0$. Specifically, set $h_j()$ as follows:

$$h_j = \exp\left[\frac{-i\zeta}{2} \left(\sum_{k=1}^{j} p_k(2T_k) + \sum_{k'=j+1}^{n} p_{k'}(T_{k'} + T_{k'}) \right) \right] \text{for} j = 0, \ldots, n,$$

$$K_{n,j} = \int d\zeta \int dp_0 \ldots dp_n \exp\left[\frac{-i}{2\epsilon} \phi_{LD} \right] h_j \tilde{f}_{LD},$$

$$L_{n,j}(t) = \frac{1}{\epsilon^d} \frac{1}{\epsilon^n} \int^{t,n} ds \int_{\Omega_n} d\Delta s K_{n,j}. \tag{3.33}$$

We have $L_n(t) = L_{n,0}(t)$. Our goal in this subsection will be to prove the following lemma.

Lemma 3.4

$$|L_{n,0}(t) - L_{n,n}(t)| \le \sqrt{\epsilon} \frac{(Ct)^{n-1}(1+t^4)}{(n-1)!} \|\psi\|_2^2 \tag{3.34}$$

Proof of Lemma 3.4: We first express $h_j - h_{j-1}$.

$$h_j - h_{j-1} = \exp\left[\frac{-i\zeta}{2} \cdot \left(\sum_{k=1}^{j-1} p_k(2T_k) + \sum_{k'=j}^{n} p_{k'}(T_{k'} + T_{k'})\right)\right] \times \left(\exp\left[\frac{-i\zeta}{2} \cdot p_j(\Delta s_j - \Delta s_{j-1})\right] - 1\right). \tag{3.35}$$

Applying Lemma 3.2 and the analysis of Lemma 3.3 we have,

$$|K_{n,j} - K_{n,j-1}| \leq C^n S_n \int d\zeta \int dp_0 \ldots dp_n \Upsilon_n(\tilde{f}_{LD}) \sum_{\alpha=0}^{4} \left| \nabla_{p_j}^{\alpha} \times \left(\exp[\frac{-i\zeta}{2} \cdot p_j(\Delta s_j - \Delta s_{j-1})] - 1\right)\right| \tag{3.36}$$

In contrast to the proof of Lemma 3.3, we have one new term to deal with. We have,

$$\begin{aligned}\sum_{\alpha=0}^{4} \left|\nabla_{p_j}^{\alpha}\left(\exp\left[\frac{-i\zeta}{2} \cdot p_j(\Delta s_j - \Delta s_{j-1})\right] - 1\right)\right| \\ \times \ \leq C(1 + |p_j|)(1 + |\zeta|^4)|(\Delta s_j - \Delta s_{j-1})|(1 + |\Delta s_j - \Delta s_{j-1}|^3). \\ \leq C(1 + t^3)(1 + |p_j|)(1 + |\zeta|^4)|(\Delta s_j - \Delta s_{j-1})|\end{aligned} \tag{3.37}$$

We then note that we can expand p_j as follows:

$$p_j = (p_j - p_{j-1}) + \cdots + (p_1 - p_0) + p_0; \tag{3.38}$$

giving

$$1 + |p_j| \leq \left(\prod_{k=1}^{j}(1 + |p_k - p_{k-1}|)\right)(1 + |p_0|). \tag{3.39}$$

Plugging this all in we have the following bound

$$|K_{n,j} - K_{n,j-1}| \leq C^n(1 + t^3)S_n|(\Delta s_j - \Delta s_{j-1})| \int d\zeta \int dp_0 \ldots dp_n \Upsilon_n(\tilde{f}_{LD}) \times \left(\prod_{k=1}^{j}(1 + |p_k - p_{k-1}|)\right)(1 + |p_0|)(1 + \zeta^4) \tag{3.40}$$

This long expression is almost the same as what we had in the proof of Lemma 3.3. One difference, seen from examining the bound for $\Upsilon_n(\tilde{f}_{LD})$ in Lemma 3.3 is that instead of integrating $\hat{R}(p_j - p_{j+1})$ we now integrate $(1 +$

$|p_j - p_{j-1}|)\hat{R}(p_j - p_{j+1})$. This will have no effect by our assumptions of $\hat{R}$. The other difference is that we include the factors $(1+\zeta^4)$ and $|(\Delta s_j - \Delta s_{j-1})|$. The $1+|\zeta|^4$ term can be controlled by the $\hat{J}$ found in the bound for $\Upsilon_n(\tilde{f}_{LD})$. So by the same arguments as in the case of Lemma 3.3 we have

$$|K_{n,j} - K_{n,j-1}| \le C^n \epsilon^d (1+t^3)|(\Delta s_j - \Delta s_{j-1})| \|\psi\|_2^2 S_n \tag{3.41}$$

We now note that in $\int_{\Omega_n} d\Delta s$ each coordinate Δs_j has its region of integration contained within $[t, -t]$. With this in mind we have

$$\begin{aligned} |L_{n,j} - L_{n,j-1}| &\le \frac{(Ct)^n}{n!}(1+t^3)\|\psi\|_2^2 \int_{\frac{-2t}{\epsilon}}^{\frac{2t}{\epsilon}} d\Delta s \frac{\epsilon \Delta s}{1+|\Delta s|^{3/2}} \\ &\le \sqrt{\epsilon}\frac{(Ct)^{n-1}(1+t^4)}{n!}\|\psi\|_2^2. \end{aligned} \tag{3.42}$$

A sum of n triangle inequalities now finishes the proof. □

3.3.3. Extension of Ω_n

In this section we will show that replacing the integration over the region Ω_n with an unbounded integration leads to a small error. To demonstrate this we introduce two new regions of integration: $\Omega_{n,1}$ and $\Omega_{n,2}$. Recall

$$\int_{\Omega_n} d\Delta s = \int_{s_1-t}^{s_1} d\Delta s_1 \int_{\Delta s_1+(s_2-s_1)}^{s_2} d\Delta s_2 \ldots \int_{\Delta s_{n-1}+(s_n-s_{n-1})}^{s_n} d\Delta s_n. \tag{3.43}$$

We set

$$\begin{aligned} \int_{\Omega_{n,1}} d\Delta s &= \int_{s_1-t}^{s_1} d\Delta s_1 \int_{(s_2-s_1)}^{s_2} d\Delta s_2 \ldots \int_{(s_n-s_{n-1})}^{s_n} d\Delta s_n; \\ \int_{\Omega_{n,2}} d\Delta s &= \int_{-\infty}^{\infty} d\Delta s_1 \int_{-\infty}^{\infty} d\Delta s_2 \ldots \int_{-\infty}^{\infty} d\Delta s_n. \end{aligned} \tag{3.44}$$

Notice that $\Omega_{n,2} = \Re^n$. We have

$$L_{n,n}(t) = \frac{1}{\epsilon^n \epsilon^d}\int^{t,n} ds \int_{\Omega_n} d\Delta s K_{n,n}, \tag{3.45}$$

and we define

$$\begin{aligned} L_n^{(1)}(t) &= \frac{1}{\epsilon^n \epsilon^d}\int^{t,n} ds \int_{\Omega_{n,1}} d\Delta s K_{n,n}, \\ L_n^{(2)}(t) &= \frac{1}{\epsilon^n \epsilon^d}\int^{t,n} ds \int_{\Omega_{n,2}} d\Delta s K_{n,n}. \end{aligned} \tag{3.46}$$

Our goal is to show that $|L_{n,n}(t) - L_n^{(1)}(t)|$ and $|L_n^{(1)}(t) - L_n^{(2)}(t)|$ are small. Before proceeding to do this we state the following lemma. The proof is straightforward.

Lemma 3.5 *Define*

$$E_{\gamma,j} = \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_{j-1} \int_{|t_j|>\gamma} dt_j \frac{1}{1+|t_1|^{3/2}} \frac{1}{1+|t_1-t_2|^{3/2}} \cdots \times \frac{1}{1+|t_{j-1}-t_j|^{3/2}}. \tag{3.47}$$

Then $E_{\gamma,j} \leq C^j \gamma^{-1/2}$.

We now bound $|L_{n,n}(t) - L_n^{(1)}(t)|$. We have the following lemma.

Lemma 3.6

$$|L_{n,n}(t) - L_n^{(1)}(t)| \leq \sqrt{\epsilon} \frac{(Ct)^{n-1}\sqrt{t}}{(n-1)!} \|\psi\|_2^2 \tag{3.48}$$

Proof of Lemma 3.6: We need to show that replacing the region Ω_n by $\Omega_{n,1}$ causes a small error. In order to demonstrate this, we introduce regions Γ_j for $j = 0, \ldots, n$ such that $\Gamma_0 = \Omega_n$ and $\Gamma_n = \Omega_{n,1}$. We will then show that replacing the region Γ_j by Γ_{j-1} causes a small error. With this in mind, set the following notation:

$$\int_{\Gamma_j} d\Delta s = \int_{s_1-t}^{s_1} d\Delta s_1 \int_{s_2-s_1}^{s_2} d\Delta s_2 \ldots \int_{s_j-s_{j-1}}^{s_j} d\Delta s_j \times \int_{\Delta s_j+(s_{j+1}-s_j)}^{s_{j+1}} d\Delta s_{j+1} \ldots \int_{\Delta s_{n-1}+(s_n-s_{n-1})}^{s_n} d\Delta s_n. \tag{3.49}$$

We also define

$$L_n^{(1,j)}(t) = \frac{1}{\epsilon^n \epsilon^d} \int^{t,n} ds \int_{\Gamma_j} d\Delta s K_{n,n}(\Delta s_1, \ldots, \Delta s_n). \tag{3.50}$$

This gives $L_n^{(1,1)}(t) = L_{n,n}(t)$ and $L_n^{(1,n)}(t) = L_n^{(1)}(t)$. We now proceed to bound $|L_n^{(1,j)}(t) - L_n^{(1,j+1)}(t)|$. We have

$$L_n^{(1,j)}(t) - L_n^{(1,j+1)}(t) = \frac{1}{\epsilon^n \epsilon^d} \int^{t,n} ds \int_{s_1-t}^{s_1} d\Delta s_1 \int_{s_2-s_1}^{s_2} d\Delta s_2 \ldots \int_{s_j-s_{j-1}}^{s_j} d\Delta s_j \times \int_{\Delta s_j+(s_{j+1}-s_j)}^{s_{j+1}-s_j} d\Delta s_{j+1} \int_{\Delta s_{j+1}+(s_{j+2}-s_{j+1})}^{s_{j+2}} d\Delta s_{j+2} \ldots \int_{\Delta s_{n-1}+(s_n-s_{n-1})}^{s_n} d\Delta s_n K_{n,n}. \tag{3.51}$$

We now apply Lemma 3.1 and the transforms $\Delta s_k \to \Delta s_k/\epsilon$ for $k = j+2, \ldots, n$. Then we have the following bound:

$$|L_n^{(1,j)}(t) - L_n^{(1,j+1)}(t)| \leq \frac{C^n}{\epsilon^{j+1}} \|\psi\|_2^2 \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_j} ds_{j+1}$$
$$\times \int_{-\infty}^{\infty} d\Delta s_1 \ldots \int_{-\infty}^{\infty} d\Delta s_j \int_{s_{j+1}-s_j}^{\Delta s_j+(s_{j+1}-s_j)} d\Delta s_{j+1} S_{j+1} \frac{(s_{j+1})^{n-(j+1)}}{(n-(j+1))!}. \quad (3.52)$$

Now split into two cases. First take $|\Delta s_j| \leq |\frac{s_{j+1}-s_j}{4}|$. Label $|L_n^{(1,j)}(t) - L_n^{(1,j+1)}(t)|$ with this restriction F_1. Then we have after transforming $\Delta s_{j+1} \to \frac{\Delta s_{j+1}}{\epsilon}$:

$$\begin{aligned}
F_1 &< \frac{C^n}{\epsilon^j} \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_j} ds_{j+1} \int d\Delta s_1 \ldots \int d\Delta s_j \\
&\quad \times \int_{\frac{s_j-s_{j+1}}{2\epsilon}}^{\frac{5(s_j-s_{j+1})}{4\epsilon}} d\Delta s_{j+1} S_j \frac{2^{3/2}}{1+\Delta s_{j+1}^{3/2}} \frac{(s_{j+1})^{n-(j+1)}}{(n-(j+1))!} \\
&\leq C^n \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_j} ds_{j+1} \frac{(s_{j+1})^{n-(j+1)}}{(n-(j+1))!} \sqrt{\frac{\epsilon}{s_j - s_{j+1}}} \\
&\leq C^n \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_j} ds_{j+1} \frac{(s_j)^{n-(j+1)}}{(n-(j+1))!} \sqrt{\frac{\epsilon}{s_{j+1}}} \\
&\leq \sqrt{\epsilon} \frac{(Ct)^{n-1}\sqrt{t}}{(n-1)!}.
\end{aligned} \quad (3.53)$$

Now take the case $|\Delta s_j| > |\frac{s_{j+1}-s_j}{4}|$. Label $|L_n^{(1,j)}(t) - L_n^{(1,j+1)}(t)|$ with this restriction F_2. Then applying Lemma 3.3.3 we have

$$\begin{aligned}
F_2 &< \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_j} ds_{j+1} \frac{(s_{j+1})^{n-(j+1)}}{(n-(j+1))!} E_{\frac{s_j-s_{j+1}}{4\epsilon}, j} \\
&\leq C^n \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_j} ds_{j+1} \frac{(s_{j+1})^{n-(j+1)}}{(n-(j+1))!} \sqrt{\frac{\epsilon}{s_j - s_{j+1}}} \\
&\leq \sqrt{\epsilon} \frac{(Ct)^{n-1}\sqrt{t}}{(n-1)!}.
\end{aligned} \quad (3.54)$$

Summing these error bounds over all j and applying the triangle inequality finishes the proof. □

Next we bound $|L_n^{(1)}(t) - L_n^{(2)}(t)|$. We have the following lemma whose proof is very similar to that of Lemma 3.6.

Lemma 3.7

$$\left|L_n^{(1)}(t) - L_n^{(2)}(t)\right| \le \sqrt{\epsilon}\frac{(Ct)^{n-1}\sqrt{t}}{(n-1)!}\|\psi\|_2^2. \tag{3.55}$$

3.3.4. Localizing Momentum Coordinates to Stationary Phase Point

We have the following formula for $L_n^{(2)}(t)$:

$$L_n^{(2)}(t) = \frac{1}{\epsilon^d}\frac{1}{\epsilon^n}\int^{t,n} ds \int_{\Re^n} d\Delta s_1 \dots d\Delta s_n \times \int d\zeta \int dp_0 \dots dp_n \exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right] h_n \tilde{f}_{LD}; \tag{3.56}$$

where we recall that

$$h_n = \exp\left[-i\zeta \cdot \left(\sum_{k=1}^{j} p_j(s_j - s_{j-1})\right)\right]. \tag{3.57}$$

In this section we prove the following lemma.

Lemma 3.8

$$L_n^{(2)}(t) = (2\pi)^{d/2}\int^{t,n} ds \int dp_n (2\pi|p_n|)^n \int dx \int^{n-1} d\hat{\mu} J(x, |p_n|\hat{\mu}_0) \times \prod_{j=0}^{n-1} \hat{R}(|p_n|\hat{\mu}_j - |p_n|\hat{\mu}_{j+1}) W_\epsilon(0, x - \sum_{k=0}^{n} |p_n|\hat{\mu}_k(s_k - s_{k+1}), p_n), \tag{3.58}$$

This lemma is fairly straightforward. All we have to do is perform an integration in each pair of Δs_j and r_{j-1}. However, there are some technical issues in doing this. We rely on the following lemma which is easily proven.

Lemma 3.9 *Let $r \in \Re^+$ and $s \in \Re$. Let $\int drr^{d-1}e^{is\phi(r)}g(r) \in L^1(\Re)$, $r^{d-1}g(r) \in L^1(\Re^+)$, $g(r) \in C(\Re^+)$, and $g(r) \in L^\infty(\Re^+)$. Assume that $\phi''(r) > a > 0$, $\phi'(r)0$, $\phi(r) = 0$ iff. $r = r_0$, and $\phi'(r_0) > 0$. Then*

$$\int_{-\infty}^{\infty} ds \int drr^{d-1}e^{is\phi(r)}g(r) = 2\pi\frac{r_0^{d-1}g(r_0)}{|\phi'(r_0)|}. \tag{3.59}$$

We now use Lemma 3.9 to prove Lemma 3.8.

Proof of Lemma 3.8: We need to show that the form of $L_n^{(2)}(t)$ is amenable to the application of Lemma 3.9. We do this in n steps. We first integrate in r_{n-1} and Δs_n, then we integrate in r_{n-2} and Δs_{n-1}, and so on down to r_0 and Δs_1. At each step we show that we can apply Lemma 3.9. The main tool in showing this is Lemma 3.2.

One issue in all this is justifying the use of Fubini. We can deal with this concern by introducing factors $e^{-\delta\Delta s_j}$ for $j = 1, \ldots, n$ on the right hand side of (3.56). If we take $\delta \to 0$ we recover $L_n^{(2)}(t)$ by Lemma 3.2. With this factor the integral is absolutely convergent and we may apply Fubini as we please.

One additional sticking point is the case of $|p_n| = 0$, but this situation can be ignored since it has Lebesgue measure zero. □

3.3.5. The Weak Limit

Up to this point the only assumption about our initial data that we have needed is its membership in $L^2(\Re^d)$. Now we would like to take the limit of W_ϵ. In order to do this and have asymptotic bounds in ϵ we depend on the WKB form of our initial data given by (1.3). Specifically we have the following lemma which is a consequence of a standard stationary phase argument.[4,6]

Lemma 3.10 *$W_\epsilon(0, x, p) = h^2(x)\delta(p - \nabla S(x)) + O(\epsilon)$ in $\mathcal{S}'(\Re^d \times \Re^d)$.*

If we define

$$\tilde{J}(x, p; \vec{s}) = (2\pi|p|)^n \int^{n-1} d\hat{\mu} J(x + \sum_{k=0}^{n} |p|\hat{\mu}_k(s_k - s_{k+1}), |p|\hat{\mu}_0) \times \prod_{j=0}^{n-1} \hat{R}(|p|\hat{\mu}_j - |p|\hat{\mu}_{j+1}), \tag{3.60}$$

then

$$L_n^{(2)}(t) = (2\pi)^{d/2} \int^{t,n} ds \left(\int dx dp \tilde{J}(x, p; \vec{s}) W_\epsilon(0, x, p) \right) \tag{3.61}$$

Notice that $\tilde{J}$ is not in $\mathcal{S}(\Re^d \times \Re^d)$ due to the singularity at $p = 0$. However, the singularity exists on a surface of dimension 3 and since we are integrating over $\Re^3 \times \Re^3$ this singularity will not effect the stationary phase arguments that make Lemma 3.10 true. So we arrive at the following lemma.

Lemma 3.11

$$\Bigg|L_n^{(2)}(t) - \Bigg((2\pi)^{d/2}\int^{t,n} ds \int dx (2\pi|\nabla S(x)|)^n \int^{n-1} d\hat{\mu}$$

$$J(x + \sum_{k=0}^{n} |\nabla S(x)|\hat{\mu}_k(s_k - s_{k+1}), |\nabla S(x)|\hat{\mu}_0)$$

$$\prod_{j=0}^{n-1} \hat{R}(|\nabla S(x)|\hat{\mu}_j - |\nabla S(x)|\hat{\mu}_{j+1})h^2(x)\Bigg)\Bigg| \leq \epsilon \frac{(Ct)^n}{n!}. \qquad (3.62)$$

4. SIMPLE DIAGRAMS

In this section we analyze simple diagrams. We first write down an explicit formula for simple diagrams. We then use our formal stationary phase assumption to write down a formal asymptotic expansion for the simple diagrams. It is possible to prove that our asymptotic expansion for simple diagrams holds rigorously, but the proof contains many details and is essentially the same as for the ladder diagram case and so we do not include it here. Details concerning the proof of the stationary phase expansion for the simple diagrams can be found in Refs. 7.

A simple diagram can be characterized by the number of ladder rungs it contains and the number of potential interactions that occur between each ladder rung. We set the following notation which uniquely describes a simple diagram.

- A = number of ladder rungs
- $j_k = \frac{1}{2}$ (the number of vertices between the kth and (k+1)th rungs in the top section of the diagram)
- $h_k = \frac{1}{2}$ (the number of vertices between the kth and (k+1)th rungs in the bottom section of the diagram)
- $M_k = j_k + h_k$
- $n = A + 2(j_0 + \cdots + j_A)$
- $n' = A + 2(h_0 + \cdots + h_A)$

For the rest of this section, we assume that the pairing π corresponds to the simple diagram specified by the above variables.

In Figure 6 we represent a generic piece of a simple diagram between two ladder rungs. Notice that the momentum and time variable indices are different than those used for ladder diagrams. The notation $p_{k,j}$ represents the jth momentum between the kth and $k + 1$th rungs of the simple diagram.

In the following sections we will make use of the notation we have developed for ladder diagrams. However, the notation for time and momentum variables introduced through Figure 6 is different than that used for ladder diagrams. The

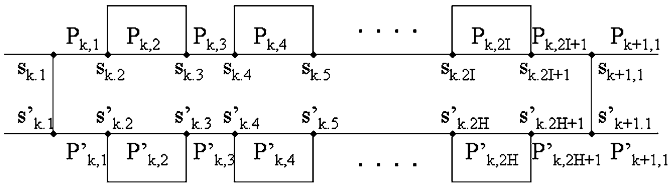

Fig. 6. Notation for Arbitrary Simple Diagram. $I = j_k$ and $H = h_k$.

following substitutions will change notation from the ladder diagram case to the simple diagram case:

$$p_{k,1} \to p_k.$$
$$s_{k,1} \to s_k.$$
$$A \to n \tag{4.1}$$

So for instance in the ladder diagram case we have $\Delta s_k = s_k - s'_k$. In the simple diagram notation this becomes $\Delta s_k = s_{k,1} - s'_{k,1}$. Similarly in the ladder diagram case we have

$$\int^{t,n} ds = \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n, \tag{4.2}$$

in the simple diagram case this notation becomes

$$\int^{t,A} ds = \int_0^t ds_{1,1} \int_0^{s_{1,1}} ds_{2,1} \ldots \int_0^{s_{A-1,1}} ds_{A,1}. \tag{4.3}$$

Or, to give another example, in the ladder diagram case we have

$$\phi_{LD}(p_0, p_1, \ldots, p_n, \Delta s_1, \ldots, \Delta s_n)$$
$$= -p_0^2 \Delta s_1 + p_1^2(\Delta s_1 - \Delta s_2) + \ldots + p_{n-1}^2(\Delta s_{n-1} - \Delta s_n) + p_n^2 \Delta s_n, \tag{4.4}$$

and in the simple diagram case this formula becomes

$$\phi_{LD}(p_{0,1}, p_{1,1}, \ldots, p_{A,1}, \Delta s_1, \ldots, \Delta s_A)$$
$$= -p_{0,1}^2 \Delta s_1 + p_{1,1}^2(\Delta s_1 - \Delta s_2) + \ldots + p_{A-1,1}^2(\Delta s_{A-1} - \Delta s_A)$$
$$+p_{A,1}^2 \Delta s_A. \tag{4.5}$$

Finally we introduce some notation that is unique to the simple diagram case. The reader is encouraged to skip this notation until it is first used in Eq. 4.8

$$\int^{(a,b),J} d\vec{s}_k = \int_a^b ds_{k,2} \int_a^{s_{k,2}} ds_{k,3} \ldots \int_a^{s_{k,2J}} ds_{k,2J+1}.$$

$$\int^{(a,b),H} d\vec{s'}_k = \int_a^b ds'_{k,2} \int_a^{s'_{k,2}} ds'_{k,3} \ldots \int_a^{s'_{k,2H}} ds'_{k,2H+1}. \tag{4.6}$$

4.1. Simple Diagram Formula

Before writing out the full formula for a simple diagram we will write out the portion of the formula corresponding to the part of the diagram between the kth and $k+1$th rung. As we did in the ladder diagram case, we first explicitely evaluate the pairings of the potential terms that correspond to the given simple diagram. However, for the moment, we only consider the pairings between the kth and $k+1$th ladder rungs. We refer to this smaller sets of pairings as π_k. We do not include in π_k the pairing corresponding to either the kth or $k+1$th ladder rung. We have then

$$\begin{aligned} E^{\pi_k} & \left[\prod_{j=1}^{2j_k} \hat{V}(p_{k,j} - p_{k,j+1}) \prod_{h=1}^{2h_k} \bar{\hat{V}}(p'_{k,h} - p'_{k,h+1}) \right] \\ &= \prod_{j=1}^{j_k} \hat{R}(p_{k,1} - p_{k,2j}) \delta(p_{k,1} - p_{k,2j+1}) \\ &\quad \times \prod_{h=1}^{h_k} \bar{\hat{R}}(p'_{k,1} - p'_{k,2h}) \delta(p'_{k,1} - p'_{k,2h+1}) \end{aligned} \tag{4.7}$$

It is then easy to show that we have the following formula for the simple diagram associated with π:

$$\begin{aligned} E^{\pi} & \left[\hat{\psi}_n\left(t, \frac{p_{0,1}}{\epsilon}\right) \bar{\hat{\psi}}_{n'}\left(t, \frac{p'_{0,1}}{\epsilon}\right) \right] \\ &= \frac{1}{\epsilon^A} \int^{t,A} ds \int^{t,A} ds' \int \prod_{j=1}^{A} dp_{j,1} \exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right] f_{LD} \prod_{k=0}^{A} F_k; \end{aligned} \tag{4.8}$$

where

$$p_{0,1} - p'_{0,1} = \zeta, \tag{4.9}$$

$$\begin{aligned} F_k &= \left(\frac{-1}{\epsilon}\right)^{j_k} \left(\frac{-1}{\epsilon}\right)^{h_k} \int^{(s_{k,1}, s_{k+1,1}), j_k} d\vec{s}_k \int^{(s'_{k,1}, s'_{k+1,1}), h_k} d\vec{s}'_k \int \prod_{j=1}^{j_k} dp_{k,2j} \\ &\quad \times \int \prod_{h=1}^{h_k} dp_{k,2h} \exp\left[\frac{-i}{2\epsilon}(\phi_{SD,k} + \epsilon\phi_{SD,k,1} + \epsilon^2\phi_{SD,k,2})\right] f_k, \end{aligned} \tag{4.10}$$

$$f_k = \prod_{j=0}^{j_k} \hat{R}\Big(p_{k,1} + \epsilon\frac{\zeta}{2} - p_{k,2j}\Big) \prod_{h=0}^{h_k} \hat{R}\Big(p_{k,1} - \epsilon\frac{\zeta}{2} - p'_{k,2h}\Big), \tag{4.11}$$

and

$$\begin{aligned} \phi_{SD,k} &= \sum_{j=1}^{j_k} \big(p_{k,2j}^2 - p_{k,1}^2\big)(s_{k,2j} - s_{k,2j+1}) \\ &\quad - \sum_{h=1}^{h_k} \big({p'_{k,2h}}^2 - {p_{k,1}}^2\big)(s'_{k,2h} - s'_{k,2h+1}), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \phi_{SD,k,1} &= -p_{k,1}\cdot\zeta \left(\sum_{j=1}^{j_k} s_{k,2j} - s_{k,2j+1} + \sum_{h=1}^{h_k} s'_{k,2h} - s'_{k,2h+1} \right), \\ \phi_{SD,k,2} &= -\frac{\zeta^2}{4} \left(\sum_{j=1}^{j_k} s_{k,2j} - s_{k,2j+1} - \sum_{h=1}^{h_k} s'_{k,2h} - s'_{k,2h+1} \right). \end{aligned} \tag{4.13}$$

Notice that our formula for the simple diagram, Eq. (4.8), is identical to that of a ladder diagram of length A except for the term $\prod F_k$.

4.2. Formal Stationary Phase Expansion for Simple Diagrams

We are interested in computing an expansion for the following integral:

$$L_{n,n'}^{SD}(t) = \int dp_{0,1} d\zeta\, \hat{J}(\zeta, p_{0,1}) \frac{1}{\epsilon^d} E^{\pi} \left[\hat{\psi}_n\Big(t, \frac{p_{0,1}}{\epsilon} + \frac{\zeta}{2}\Big) \bar{\hat{\psi}}_{n'}\Big(t, \frac{p_{0,1}}{\epsilon} - \frac{\zeta}{2}\Big) \right]. \tag{4.14}$$

Plugging in our formula for a simple diagram pairing, Eq. (4.8), gives the following:

$$\begin{aligned} L_{n,n'}^{SD}(t) &= \int dp_{0,1} d\zeta\, \hat{J}(\zeta, p_{0,1}) \frac{1}{\epsilon^d} \frac{1}{\epsilon^A} \int^{t,A} ds \int^{t,A} ds' \\ &\quad \times \int \prod_{j=1}^{A} dp_{j,1} \exp\Big[\frac{-i}{2\epsilon}\phi_{LD}\Big] f_{LD} \prod_{k=0}^{A} F_k. \end{aligned} \tag{4.15}$$

We compute a stationary phase expansion for $L_{n,n'}^{SD}$ formally. We do this in three steps. First, we find a stationary phase expansion for each F_k. We use this expansion to write down an expansion for $\prod_{k=0}^{A} F_k$ that allows us to reduce the simple diagram into a ladder diagram. Finally we apply our ladder diagram analysis to the reduced simple diagram.

4.2.1. Stationary Phase Expansion for F_k and $\prod_{k=0}^{A} F_k$

In order to compute F_k, we first find the stationary phase surface of $\phi_{SD,k}$. We introduce the variables $r_{k,2j}, r'_{k,2h}, \hat{\mu}_{k,2j}, \hat{\mu}'_{k,2h}$ such that $p_{k,2j} = r_{k,2j}\hat{\mu}_{k,2j}$ and $p'_{k,2h} = r'_{k,2h}\hat{\mu}'_{k,2h}$. We also introduce the time variables $\Delta s_{k,2j} = s_{k,2j} - s_{k,2j+1}$ and $\Delta s'_{k,2h} = s_{k,2h} - s'_{k,2h+1}$. Then our stationary phase surface is specified by the following constraints:

$$\begin{aligned} r_{k,2j} &= |p_{k,1}| \quad \text{for } j = 1, \ldots, j_k; \\ r'_{k,2h} &= |p_{k,1}| \quad \text{for } h = 1, \ldots, h_k; \\ \Delta s_{k,2j} &= 0 \qquad \text{for } j = 1, \ldots, j_k; \\ \Delta s'_{k,2h} &= 0 \qquad \text{for } h = 1, \ldots, h_k. \end{aligned} \tag{4.16}$$

Now we want to apply a formal stationary phase argument in these new variables. First we rewrite the integral $\int^{(s_{k,1},s_{k+1,1}),j_k} d\vec{s}_k$ in terms of our new coordinates.

$$\int^{(s_{k,1},s_{k+1,1}),j_k} d\vec{s}_k = \int_{s_{k+1,1}}^{s_{k,1}} ds_{k,2} \int_{s_{k+1,1}}^{s_{k,2}} ds_{k,4} \ldots \int_{s_{k+1,1}}^{s_{k,2j_k-2}} ds_{k,2j_k}$$
$$\times \left(\int_0^{s_{k,2}-s_{k,4}} d\Delta s_{k,2} \int_0^{s_{k,4}-s_{k,6}} d\Delta s_{k,4} \ldots \int_0^{s_{k,2j_k}-s_{k+1,1}} d\Delta s_{k,2j_k} \right) \tag{4.17}$$

A similar formula holds for $\int^{(s'_{k,1},s'_{k+1,1}),h_k} d\vec{s}'_k$.

The above formula reveals that the stationary phase points of $\phi_{SD,k}$ are always on the boundary of the region of integration! Hence we cannot apply our formal stationary phase expansion. However we may alter the boundary to make the computation of F_k straightforward.

Our idea is to change the formula for F_k by replacing the region of integration given by $\int^{(s_{k,1},s_{k+1,1}),j_k} d\vec{s}_k$ by a region Γ_k defined as follows:

$$\int_{\Gamma_k} d\vec{s}_k = \int_{s_{k+1,1}}^{s_{k,1}} ds_{k,2} \int_{s_{k+1,1}}^{s_{k,2}} ds_{k,4} \ldots \int_{s_{k+1,1}}^{s_{k,2j_k-2}} ds_{k,2j_k}$$
$$\times \left(\int_0^{t} d\Delta s_{k,2} \int_0^{t} d\Delta s_{k,4} \ldots \int_0^{t} d\Delta s_{k,2j_k} \right) \tag{4.18}$$

We similarly define regions Γ'_k to replace the regions in $\int^{(s'_{k,1},s'_{k+1,1}),h_k} d\vec{s}'_k$. Then we define $\tilde{F}_k$ as the altered form of F_k. That is,

$$\tilde{F}_k = \left(\frac{-1}{\epsilon}\right)^{j_k} \left(\frac{-1}{\epsilon}\right)^{h_k} \int_{\Gamma_k} d\vec{s}_k \int_{\Gamma'_k} d\vec{s}'_k \int \prod_{j=1}^{j_k} dp_{k,2j} \int \prod_{h=1}^{h_k} dp_{k,2h}$$
$$\times \exp\left[\frac{-i}{2\epsilon}(\phi_{SD,k} + \epsilon\phi_{SD,k,1} + \epsilon^2\phi_{SD,k,2})\right] f_k, \tag{4.19}$$

If we consider $F_k - \tilde{F}_k$, no stationary point of $\phi_{SD,k}$ exists in the resultant region of integration. It follows easily from our formal stationary phase expansion that we can bound $|F_k - \tilde{F}_k|$ as follows:

$$|F_k - \tilde{F}_k| \leq C^{j_k+h_k}\sqrt{\epsilon}\frac{(s_{k,1} - s_{k+1,1})^{j_k}}{j_k!}\frac{(s'_{k,1} - s'_{k+1,1})^{h_k}}{h_k!} \tag{4.20}$$

The point of all this is that $\tilde{F}_k$ is trivial to compute.

$$\tilde{F}_k = (-1)^{j_k+h_k}\frac{\Sigma_1\left(p_{k,1}, t, \zeta, \epsilon\frac{\zeta}{2}\right)^{j_k}(s_{k,1} - s_{k+1,1})^{j_k}}{j_k!} \times \frac{\bar{\Sigma}_1\left(p_{k,1}, t, \zeta, -\epsilon\frac{\zeta}{2}\right)^{h_k}(s'_{k,1} - s'_{k+1,1})^{h_k}}{h_k!}; \tag{4.21}$$

where

$$\Sigma_1(p, t, \zeta, \eta) \times \int_0^{\frac{t}{\epsilon}} ds \int dq \exp\left[\frac{-is}{2}(q^2 - p^2) - \epsilon^2 p \cdot \zeta s - \epsilon^3\frac{\zeta^2}{4}s\right]\hat{R}(p + \eta - q). \tag{4.22}$$

We are not quite done with $\tilde{F}_k$ because the Σ_1 terms have ϵ, t, and ζ dependence which we would like to remove. This dependence can easily be removed through a taylor series expansion and an application of Lemma 3.2. Set

$$\Sigma_0(p) = \int_0^\infty ds \int dq \exp\left[\frac{-is}{2}(q^2 - p^2)s\right]\hat{R}(p - q). \tag{4.23}$$

Then

$$\left|\Sigma_0(p) - \Sigma_1\left(p, t, \zeta, \epsilon\frac{\zeta}{2}\right)\right| \leq C\sqrt{\frac{\epsilon}{t}}(1 + |\zeta|^2)(1 + |p|). \tag{4.24}$$

We can now write a very compact expansion for F_k.

$$\left|F_k - F_{k,\text{lim}}\right| \leq C(1 + |\zeta|^2)(1 + |p|)\sqrt{\frac{\epsilon}{t}}F_{k,\text{error}} \tag{4.25}$$

where

$$F_{k,\text{lim}} = (-1)^{M_k}\frac{(s_{k,1} - s_{k+1,1})^{j_k}}{j_k!}\frac{(s'_{k,1} - s'_{k+1,1})^{h_k}}{h_k!}\Sigma_0(p_{k,1})^{j_k}\bar{\Sigma}_0(p_{k,1})^{h_k},$$

$$F_{k,\text{error}} = C^{M_k}\frac{(s_{k,1} - s_{k+1,1})^{j_k}}{(j_k - 1)!}\frac{(s'_{k,1} - s'_{k+1,1})^{h_k}}{(h_k - 1)!}. \tag{4.26}$$

We now turn to finding an expansion of $\prod_{k=0}^{A} F_k$. We cannot just write this as a product of expansions of each F_k because then we would have terms of order ζ^A which we would be unable to control. Instead we simply note that rather than applying our stationary phase arguments to each F_k separately we may apply the arguments to all the F_k at once. Then a series of triangle inequalities allows us to replace each Σ_1 by Σ_0 and arrive at the following bound:

$$\left|\prod_{k=0}^{A} F_k - \prod_{k=0}^{A} F_{k,\mathrm{lim}}\right| \le C(1+|\zeta|^2)\left(\sum_{k=0}^{A} 1+|p_{k,1}|\right)\sqrt{\frac{\epsilon}{t}}\prod_{k=0}^{A} F_{k,\mathrm{error}}. \tag{4.27}$$

4.2.2. Reduction of Simple Diagram to Ladder Diagram

If we use the approximation of $\prod_{k=0}^{A} F_k$ given by (4.27) in the expression for $L_{n,n'}^{SD}(t)$ given by (4.15) we arrive at

$$\begin{aligned} &L_{n,n'}^{SD}(t) \\ &\quad = \int dp_{0,1}d\zeta\,\hat{J}(\zeta,p_{0,1})\frac{1}{\epsilon^d}\frac{1}{\epsilon^A}\int^{t,A} ds\int^{t,A} ds'\int\prod_{j=1}^{A} dp_{j,1}\exp\left[\frac{-i}{2\epsilon}\phi_{LD}\right] \\ &\qquad \times f_{LD}\left(\prod_{k=0}^{A} F_{k,\mathrm{lim}} + \Theta(1+\zeta^2)\Big(\sum_{k=0}^{A} 1+|p_{k,1}|\Big)\sqrt{\epsilon}\prod_{k=0}^{A} F_{k,\mathrm{error}}\right); \end{aligned} \tag{4.28}$$

where Θ is defined to satisfy

$$\prod_{k=0}^{A} F_k - \prod_{k=0}^{A} F_{k,\mathrm{lim}} = \Theta(1+|\zeta|^2)\left(\sum_{k=0}^{A} 1+|p_{k,1}|\right)\sqrt{\frac{\epsilon}{t}}\prod_{k=0}^{A} F_{k,\mathrm{error}}. \tag{4.29}$$

Notice that by (4.27), $|\Theta| \le C$.

The formulas directly above demonstrate that we have essentially reduced the simple diagram into a ladder diagram of length A; the only difference being the inclusion of the terms $F_{k,\mathrm{lim}}$ and $F_{k,\mathrm{error}}$. We would now like to apply our ladder diagram analysis to $L_{n,n'}^{SD}(t)$, however the Θ term prevents us from analyzing the remainder term.

In order to proceed, we can instead of performing a stationary phase analysis on $\prod_{k=0}^{A} F_k$ and then a stationary phase analysis on the resulting ladder diagram simply perform a single higher dimensional stationary phase analysis on the whole simple diagram. If we do this we can arrive at the following expansion for $L_{n,n'}^{SD}(t)$. Set $L = M_0 + \ldots + M_A$. Let $\delta_{k,j}$ be the Kronecker delta function.

$$\Bigg| L^{SD}_{n,n'}(t) - (2\pi)^{d/2} \int^{t,A} ds \int dp_{A,1} \int dx \int^{n} d\hat{\mu} J(x, |p_{A,1}|\hat{\mu}_0) \Gamma_A W_\epsilon(0, x_0, p_{A,1}) \Bigg| \leq \sqrt{\epsilon} \frac{(Ct)^{A+L-1}\sqrt{t}}{(A+L)!} \prod_{k=0}^{A} \frac{M_k!}{j_k! h_k!} (1 - \delta_{A+L,0}); \tag{4.30}$$

with

$$\Gamma_A(\hat{\mu}, p_A, s_1, \ldots, s_n) = \left(\prod_{k=0}^{A} (-1)^{M_k} (\Sigma_0(p_A)^{j_k} (\bar{\Sigma}_0(p_A))^{h_k} \frac{(s_k - s_{k+1})^{M_k}}{j_k! h_k!} \right) \times \left(\prod_{k'=0}^{A-1} \sigma(|p_A|\hat{\mu}_k, |p_A|\hat{\mu}_{k+1}) \right), \tag{4.31}$$

$$x_0 = x - \sum_{m=0}^{A} |p_{A,1}|\hat{\mu}_m (s_{m,1} - s_{m+1,1}) \tag{4.32}$$

The above formulas require some explanation. First, in the case $A + L = 0$, Eq. (3.8) shows that our stationary phase approximation is exact. This explains the presence of the Kronecker delta function term $(1 - \delta_{A+L,0})$ on the right hand side of (4.30). Second, we have used the following identity to simplify the error term:

$$\int^{t,A} ds \prod_{k=0}^{A} \frac{(s_{k,1} - s_{k+1,1})^{M_k}}{j_k! h_k!} = \frac{t^{A+L}}{(A+L)!} \prod_{k=0}^{A} \frac{M_k!}{j_k! h_k!}. \tag{4.33}$$

Besides writing an expansion for each simple diagram, we can also find a bound for each simple diagram. We have the following lemma which will be useful to us in the following section.

Lemma 4.1

$$\left| L^{SD}_{n,n'}(t) \right| \leq \frac{(C(1+t))^{A+L}}{(A+L)!} \prod_{k=0}^{A} \frac{M_k!}{j_k! h_k!}. \tag{4.34}$$

Proof of Lemma 4.1: We have

$$\left| L^{SD}_{n,n'}(t) \right| \leq C^A (\sup \int^{t,A} ds |\Gamma|) \sup_{\vec{s},\vec{\hat{\mu}}} \left| \int dp_{A,1} \int dx J(x, |p_{A,1}|\hat{\mu}_0) W_\epsilon(x_0, p_{A,1}) \right| + \sqrt{\epsilon} \frac{(Ct)^{A+L-1}\sqrt{t}}{(A+L)!} \prod_{k=0}^{A} \frac{M_k!}{j_k! h_k!} (1 - \delta_{A+L,0}). \tag{4.35}$$

Then we have the following bound which follows from Eq. 4.33:

$$\sup \int^{t,A} ds|\Gamma| \leq \frac{(Ct)^{A+L}}{(A+L)!} \prod_{k=0}^{A} \frac{M_k!}{j_k! h_k!} \tag{4.36}$$

The proof is finished with the following bound,

$$\begin{aligned} &|\sup_{\vec{s},\vec{\hat{\mu}}} \int dp_{A,1} \int dx J(x, |p_{A,1}|\hat{\mu}_0) W_\epsilon(x_0, p_{A,1})| \\ &\leq \sup_{\vec{s},\vec{\hat{\mu}}} \int dp_{A,1} \int d\zeta |\hat{J}(\zeta, |p_n|\hat{\mu}_0)| \frac{1}{\epsilon^d} \left| \hat{\psi}\left(\frac{p_{A,1}}{\epsilon} - \frac{\zeta}{2}\right) \bar{\hat{\psi}}\left(\frac{p_{A,1}}{\epsilon} + \frac{\zeta}{2}\right) \right| \leq C\|\psi\|_2^2 \end{aligned} \tag{4.37}$$

□

5. PROOF OF MAIN THEOREM

Recall that we are interested in an expansion for

$$\int dpdz J(z,p) \frac{1}{\epsilon^d} E\left[W_{\psi(t)}\left(\frac{p}{\epsilon}, z\right)\right] = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \tag{5.1}$$

where the $\mathcal{L}$ terms are defined by (2.13). Our stationary phase expansions coupled with some bounds developed in Refs. 3 will allow us to control each of the $\mathcal{L}$ terms.

In this section, in order to compactify our formulas, we use the following notation similar to Refs. 3:

$$\tilde{C}^{\pi}_{n,n'} = E^{\pi}\left[\hat{\psi}_n\left(t, \frac{p}{\epsilon} + \frac{\zeta}{2}\right) \bar{\hat{\psi}}_{n'}\left(t, \frac{p}{\epsilon} - \frac{\zeta}{2}\right)\right]. \tag{5.2}$$

We start by considering $\mathcal{L}_1$. We expand $\mathcal{L}_1$ as follows:

$$\mathcal{L}_1 = \mathcal{L}_{1,\text{SD}} + \mathcal{L}_{1,\text{NSD}} \tag{5.3}$$

where

$$\begin{aligned} \mathcal{L}_{1,\text{SD}} &= \frac{1}{(2\pi)^{d/2}} \sum_{n=0}^{N} \sum_{n'=0}^{N} \sum_{\pi=\text{simple}} \int dpd\zeta \bar{\hat{J}}(\zeta, p) \frac{1}{\epsilon^d} \tilde{C}^{\pi}_{n,n'}; \\ \mathcal{L}_{1,\text{NSD}} &= \frac{1}{(2\pi)^{d/2}} \sum_{n=0}^{N} \sum_{n'=0}^{N} \sum_{\pi\neq\text{simple}} \int dpd\zeta \bar{\hat{J}}(\zeta, p) \frac{1}{\epsilon^d} \tilde{C}^{\pi}_{n,n'}. \end{aligned} \tag{5.4}$$

We can use our simple diagram expansion to prove the following lemma.

Lemma 5.1

$$
\begin{aligned}
&|\mathcal{L}_{1,\mathrm{SD}} - \int dpdx J(x,p)B(x,p)| \\
&\quad \le C\sqrt{\epsilon t}\exp[Ct\exp[Ct]] + \exp[C(1+t)]\frac{(C(1+t^2))^N}{N!}.
\end{aligned} \tag{5.5}
$$

Proof of Lemma 5.1: We can rewrite $\mathcal{L}_{1,\mathrm{SD}}$ in terms of our simple diagram notation. Recall $M_k = j_k + h_k$ and $L = M_0 + M_1 + \ldots + M_A$.

$$
\mathcal{L}_{1,\mathrm{SD}} = \underbrace{\sum_A \sum_{M_0} \cdots \sum_{M_A} \sum_{j_0} \cdots \sum_{j_A}}_{A+j_0+\ldots+j_A \le N, A+h_0+\ldots+h_A \le N} \int dpd\zeta \bar{\hat{J}}(\zeta,p) \frac{1}{\epsilon^d} \tilde{C}^{\pi}_{n,n'}; \tag{5.6}
$$

where the pairing π is the simple diagram specified by the variables in the external sum. We can now plug in our stationary phase expansion for each simple diagram, equation (4.30).

$$
\begin{aligned}
&\mathcal{L}_{1,\mathrm{SD}} \\
&= \underbrace{\sum_A \sum_{M_0} \cdots \sum_{M_A} \sum_{j_0} \cdots \sum_{j_A}}_{A+j_0+\ldots+j_A \le N, A+h_0+\ldots+h_A \le N} \Bigg(\int^{t,A} ds \int dp_{A,1} \int dx \int^n d\hat{\mu} J(x, |p_{A,1}|\hat{\mu}_0) \\
&\times \Gamma_A W_\epsilon(0, x_0, p_{A,1}) + \sqrt{\epsilon} \frac{(Ct)^{A+L-1}\sqrt{t}}{(A+L)!} \prod_{k=0}^{A} \frac{M_k!}{j_k! h_k!} (1 - \delta_{A+L,0}) \Bigg)
\end{aligned} \tag{5.7}
$$

We would now like to remove the restrictions $A + j_0 + \ldots + j_A \le N$ and $A + j_0 + \ldots + j_A \le N$ from the summation above and allow the simple diagram variables to range over all possible values. Through Lemma 4.2.2 it is fairly easy to bound the error caused by replacing the restricted sum by an unrestricted sum. Specifically we can express $\mathcal{L}_{1,\mathrm{SD}}$ as follows:

$$
\mathcal{L}_{1,\mathrm{SD}} = \mathcal{L}_{1,\mathrm{SD,lim}} + \mathcal{L}_{1,\mathrm{SD,error1}} + \mathcal{L}_{1,\mathrm{SD,error2}}; \tag{5.8}
$$

where

$$
\begin{aligned}
\mathcal{L}_{1,\mathrm{SD,lim}} &= \sum_{A=0}^{\infty} \sum_{M_0=0}^{\infty} \cdots \sum_{M_A=0}^{\infty} \sum_{j_0=0}^{M_0} \cdots \sum_{j_A=0}^{M_A} \int^{t,A} ds \int dp_{A,1} \int dx \int^n d\hat{\mu} \\
&\quad \times J(x, |p_{A,1}|\hat{\mu}_0) \Gamma_A W_\epsilon(0, x_0, p_{A,1});
\end{aligned} \tag{5.9}
$$

and where we have the following bounds for $\mathcal{L}_{1,\mathrm{SD},\mathrm{error1}}$ and $\mathcal{L}_{1,\mathrm{SD},\mathrm{error2}}$. The bound for $\mathcal{L}_{1,\mathrm{SD},\mathrm{error2}}$ is achieved by applying Lemma 4.1.

$$\mathcal{L}_{1,\mathrm{SD},\mathrm{error1}}$$
$$= \underbrace{\sum_{A}\sum_{M_0}\cdots\sum_{M_A}\sum_{j_0}\cdots\sum_{j_A}}_{A+j_0+\cdots+j_A\le N,\, A+h_0+\cdots+h_A\le N} \sqrt{\epsilon}\frac{(Ct)^{A+L-1}\sqrt{t}}{(A+L)!}\prod_{k=0}^{A}\frac{M_k!}{j_k!h_k!}(1-\delta_{A+L,0}),$$

$$\mathcal{L}_{1,\mathrm{SD},\mathrm{error2}} \le \underbrace{\sum_{A}\sum_{M_0}\cdots\sum_{M_A}\sum_{j_0}\cdots\sum_{j_A}}_{A+M_0+\cdots+M_A\ge N} \frac{(C(1+t))^{A+L}}{(A+L)!}\prod_{k=0}^{A}\frac{M_k!}{j_k!h_k!}. \tag{5.10}$$

We can easily derive a bound for $\mathcal{L}_{1,\mathrm{SD},\mathrm{error1}}$. Using the simple bound $(A+L)! \ge A!M_0!M_1!\cdots M_A!$ we have

$$\begin{aligned}
\mathcal{L}_{1,\mathrm{SD},\mathrm{error1}} &\le \sum_{A=0}^{\infty}\sum_{M_0=0}^{\infty}\cdots\sum_{M_A=0}^{\infty}\sqrt{\epsilon}\frac{2^{A+1}(Ct)^{A+L-1}\sqrt{t}}{(A+L)!}(1-\delta_{A+L,0})\\
&\le \sum_{A=0}^{\infty}\sqrt{\epsilon}\frac{2^{A+1}(Ct)^{A-1}\sqrt{t}}{(A)!}\left(\sum_{M=0}^{\infty}\frac{(Ct)^M}{(M)!}\right)^{A+1}(1-\delta_{A+L,0})\\
&\le C\sqrt{\epsilon t}\exp[Ct\exp[Ct]].
\end{aligned} \tag{5.11}$$

A bound for $\mathcal{L}_{1,\mathrm{SD},\mathrm{error2}}$ requires a bit more work. We first note that a simple inductive argument gives the following bound:

$$\underbrace{\sum_{M_0}\cdots\sum_{M_A}}_{M_0+\ldots+M_A=L} 1 \le \frac{(L+A)^A}{A!} \tag{5.12}$$

Using this bound and Stirling's formula, we have the following bound for $\mathcal{L}_{1,\mathrm{SD},\mathrm{error2}}$:

$$\mathcal{L}_{1,\mathrm{SD},\mathrm{error2}} \le \underbrace{\sum_{A}\sum_{L}}_{A+L\ge N}\frac{(C(1+t))^{A+L}}{(A+L)^L A^A} \tag{5.13}$$

We split the sum into three pieces.

$$\underbrace{\sum_{A}\sum_{L}}_{A+L\geq N} = \sum_{A=N}^{\infty}\sum_{L=0}^{\infty} + \sum_{A=0}^{N}\sum_{L=N}^{\infty} + \sum_{A=0}^{N}\sum_{L=N-A}^{N}. \tag{5.14}$$

We bound the contribution of each of these three sums. For the first sum, we have the following bound.

$$\begin{aligned}\sum_{A=N}^{\infty}\sum_{L=0}^{\infty}\frac{(C(1+t))^{A+L}}{(A+L)^L A^A} &\leq \sum_{A=N}^{\infty}\frac{(C(1+t))^A}{A^A}\exp[C(1+t)] \\ &\leq \exp[C(1+t)]\frac{(C(1+t))^N}{N!}.\end{aligned} \tag{5.15}$$

The second sum has the same bound as the first sum. Finally for the third sum we have,

$$\begin{aligned}\sum_{A=0}^{N}\sum_{L=N-A}^{N}\frac{(C(1+t))^{A+L}}{(A+L)^L A^A} &\leq N^2(C(1+t))^{2N}\sup_{A+L=N}\frac{1}{N^L A^A} \\ &= N^2(C(1+t))^{2N}\sup_{A\leq N}\frac{1}{N^N N^{-A} A^A} \\ &= N^2(C(1+t))^{2N}\frac{1}{N^N} \\ &\leq \frac{(C(1+t^2))^N}{N!}\end{aligned} \tag{5.16}$$

So finally we arrive at the following bound for $\mathcal{L}_{1,\text{SD},\text{error2}}$:

$$\mathcal{L}_{1,\text{SD},\text{error2}} \leq \exp[C(1+t)]\frac{(C(1+t^2))^N}{N!}. \tag{5.17}$$

We have left the task of computing $\mathcal{L}_{1,\text{SD},\text{lim}}$. We can bring all the summations except for that depending on A inside the integral. Recall the definition of U_A given by formula 2.3. Then

$$\sum_{M_0=0}^{\infty}\cdots\sum_{M_A=0}^{\infty}\sum_{j_0=0}^{M_0}\cdots\sum_{j_A=0}^{M_A}\Gamma_A = U_A, \tag{5.18}$$

where we have used the following equality which follows from Lemma 3.9:

$$2\text{Re}(\Sigma_0(p)) = \Sigma(p). \tag{5.19}$$

Now if we apply Lemma 3.10 to evaluate the weak limit of W_ϵ we have

$$\mathcal{L}_{1,\mathrm{SD,lim}} = \sum_{A=0}^{\infty}(J, B_A) + C\epsilon = (J, B) + C\epsilon. \tag{5.20}$$

□

At this point we need to introduce some results of Erdös and Yau.[3] Up to now we have kept track of t in all our bounds. However for the rest of this paper we will absorb t into the generic constant C. We do this for purposes of clarity, tracking t is not difficult but it produces unreadable formulas that obscure the core result.

In order to control $\mathcal{L}_{1,\mathrm{NSD}}$ we use a lemma of Erdös and Yau. The following lemma is lemmas 3.5, 3.6, and 4.1 in Refs. 3.

Lemma 5.2

$$\int dpd\zeta\,\bar{\hat{J}}(\zeta, p)\frac{1}{\epsilon^d}\tilde{C}^{\pi}_{n,n'} \le \epsilon C^N |\log\epsilon|^{N+3} \mathit{for}\,\pi = \mathit{non-simple\ pairing}. \tag{5.21}$$

$$|\mathcal{L}_{1,\mathrm{NSD}}| \le \epsilon N^2 N! C^N |\log\epsilon|^{N+3}. \tag{5.22}$$

In order to control $\mathcal{L}_2$ and $\mathcal{L}_3$ we need to bound Ψ_N. The following lemma is a slightly improved version of Lemma 5.2 in Refs. 3. We achieve the slight improvement by substituting our Lemma 4.1 for Lemma 3.7 in Refs. 3.

Lemma 5.3

$$E[\|\Psi_N\|_2^2] \le \frac{C^N N^2\kappa^2}{N!} + C^N N^2\kappa^2|\log\epsilon|^{4N+5}\epsilon(4N)! + \frac{C^N|\log\epsilon|^{4N+5}(4N)!}{\epsilon^2\kappa^N} \tag{5.23}$$

where we may choose any positive value for κ.

We are finally ready to prove our Theorem 2. This proof closely follows arguments found in Refs. 3.

Proof of Theorem 2: Our basic task in this proof will be to choose N and then apply the lemmas above to bound $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_1 - \mathcal{L}_{1,\mathrm{SD,lim}}$.

Set N and κ as follows:

$$N = \frac{1}{9}\frac{|\log\epsilon|}{\log|\log\epsilon|}.$$

$$\kappa = |\log\epsilon|^{36}. \tag{5.24}$$

Note the following bounds:

$$N! \le C_\delta \left(\frac{1}{\epsilon}\right)^{1/9-\delta},$$
$$|\log\epsilon|^N = \left(\frac{1}{\epsilon}\right)^{1/9},$$
$$\kappa^N = \left(\frac{1}{\epsilon}\right)^{4}. \tag{5.25}$$

We first turn to bounding $\mathcal{L}_1$. Applying Lemmas 5.1 and 5.2 we have

$$|\mathcal{L}_{1,\mathrm{SD}} - (J, B)| \le \sqrt{\epsilon} + \frac{C^N}{N!} \le \sqrt{\epsilon} + C_\delta \epsilon^{1/9-\delta}, \tag{5.26}$$

and

$$|\mathcal{L}_{1,\mathrm{NSD}}| \le \epsilon N^2 C^N |\log\epsilon|^{N+3} N! \le C_\delta \epsilon^{7/9-\delta}. \tag{5.27}$$

Putting the two bounds above together gives us the following error bound for $\mathcal{L}_1$:

$$|\mathcal{L}_1 - (J, B)| \le \sqrt{\epsilon} + C_\delta \epsilon^{1/9-\delta}. \tag{5.28}$$

Now we turn our attention to $\mathcal{L}_2$.

$$|\mathcal{L}_2| \le C \sum_{j=0}^{N-1} \left(E\left[\|\hat{\psi}_j(t,p)\|_2^2\right]\right)^{1/2} \left(E\left[\|\Psi_N(t,p)\|_2^2\right]^{1/2}\right)^{1/2}; \tag{5.29}$$

where we have applied Schwartz twice. From Lemma 5.2 and Lemma 4.1 we have the bound

$$\sum_{j=0}^{N-1} \left(E\left[\|\hat{\psi}_j(t,p)\|_2^2\right]\right)^{1/2} \le C + \epsilon N^3 C^N |\log\epsilon|^{N+3} N! \le C. \tag{5.30}$$

From Lemma 5.3 we have the bound

$$\begin{aligned} E[\|\Psi_N(t,p)\|_2^2] \le{}& \frac{C^N N^2 \kappa^2 |\log\epsilon|^4}{N!} + C^N N^2 \kappa^2 |\log\epsilon|^{4N+5} \epsilon (4N)! \\ &+ \frac{C^N |\log\epsilon|^{4N+5} (4N)!}{\epsilon^2 \kappa^N} \\ \le{}& C_\delta \epsilon^{1/9-\delta}. \end{aligned} \tag{5.31}$$

Using the two bounds above we have the following bound for $\mathcal{L}_2$.

$$|\mathcal{L}_2| \le C_\delta \epsilon^{1/18-\delta}. \tag{5.32}$$

Finally we bound $\mathcal{L}_3$.

$$|\mathcal{L}_3| \leq E[\|\Psi_N\|_2^2] \leq C_\delta \epsilon^{1/9-\delta}. \tag{5.33}$$

Summing the bounds for $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ finishes the proof. □

ACKNOWLEDGMENTS

I would like to thank Jonathan Goodman for his many comments, fixes, and observations. Most specifically, the original idea of trying to analyze the weak coupling problem through stationary phase methods was his. I also thank an anonymous reviewer. This work was greatly improved as a result of this reviewer's care, diligence, and knowledge.

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